A Study in Binary Relations for Logic Algebra Functions

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Abstract — **Solving the Design for Test (DFT) problem for digital circuits (DC) requires conducting complex studies of the phenomena situated at the boundaries of scientific directions, such as Digital Signal Theory, Logic and Boolean Algebra, DC Analysis and Synthesis, DC Design Fundamentals, Testing Algorithms Design, Relational Models, DC Testing. Half a century, from the moment the DFT for DC problem was raised, no concluding results were obtained, which confirms the necessity for introducing a new DFT paradigm, based on existence principles found in nature. Creating this new DFT paradigm requires the analysis, design and usage of certain binary relations between logical functions (LF), determined on the same definition domain as binary arguments tuples. This fact leads to the creation of new elementary structures and the development of a modern and efficient DFT theory.**

The present study originality lies in performing for the first time an exhaustive analysis of the logic functions binary relations in logic algebra, with the purpose of highlighting their properties, important for formalizing synthesis algorithms and elaborating the concepts for digital structures and for the indispensable know-how regarding obtaining an adequate solution for the Designed for Test problem for digital circuits.

Index Terms — **complement, duality, equivalence, partial complement.**

I. INTRODUCTION

I.1. Basic notions and definitions

In the paper we use the notions, basic definitions and notations from [1]. However, the fact that the area of logical functions (LF) used in logic algebra is larger, leads to the apparition of new LF couples, which cannot be realized with the Boolean operators AND, OR, NOT. This particularity provides new properties to LF relations in logic algebra, in establishing useful properties of LF relations, which can be utilized for elaborating a new platform of elemental digital structures, as a fundament to DFT. The purpose of this paper is the study, analysis and comparing of and fundamental logical properties of LF (F_i, F_j), determined on the same definition domain, in order to establish relevant properties for them for the purpose of elaborating a new DFT paradigm.

I.2. Logic algebra

The algebra formed form the set B = {0, 1} together with all the possible operations in this set is called logic algebra (LA). A function $f(x_1,..., x_i,..., x_n)$ is said to be contained in LA (or logic function) if it, together with its arguments $x_i, i \in \overline{1,n}$, takes values from the set B= {0,1} [2].

Boolean algebra (BA) is an important subset of logic algebra: more specific BA is most frequently used for representing LF and conducting minimization with logic redundancy exclusion and initial form synthesis of LF, based on the operators of simple Boolean base. (SBB). As required, the LF, at the next synthesis steps, will be modified maintaining logical equivalence to be represented in mono-functional universal base (M-FUB) AND-NOT or OR-NOT. Sometimes it may be required to synthesis in the mixed base (MB) of logic operators which contains operators from SBB and M-FUB or extended base (EB), which contains operators from mixed base and the LF implication, parity and/or imparity.

II. THE FUNCTION OF LOGIC ALGEBRA

In table 1 we present the 2 variable LF $B(n) = 2^{2^n} = 2^{2^2} = 16$, whose definition domain is the ordered set of tuples $X_k = (x_0^\sigma, x_1^\sigma, ..., x_{n-1}^\sigma)$, σ ∈ (0, 1), $k = \overline{0 \div 2^n - 1}$, and $F_j = F_0, F_{1,...,} F_{k,...,} F_{2^n-1}$ is the LF value domain. Here k represents the order number of the given tuple X_k, to which corresponds the respective LF $y_k=f(x_k)$, $k = \overline{0 \div 2^n - 1}$. Therefore there exists a two-way relation between the values of a tuple and the respective values of the LF:

$$y_k = f_k(x_0,...,x_i,...,x_{n-1}), \quad k = \overline{0 \div 2^n - 1}, \quad x_i \in \{0, 1\}$$
and $y_k \in \{0, 1\}$ (1)

Following, we will study the two-way relations between Boolean functions (BF) couples (F_i, F_j), which have the same domain definition.

Discovering LF relations and, especially, LF interactions, permits the scientific based determination of the elaboration of the elemental digital structure base, new logic-algebraic synthesis concepts and, finally, adequately solving the DFT problem. The relations are particular cases of the sets for which the same operations can be used.

The LF analysis from table 1 and their complemented functions shows the following types of LF relations:

1) equivalence: $F_1 / \overline{F}_{14}$, $F_7 / \overline{F}_8$, $F_6 / \overline{F}_9$;
2) complement: F_1 / F_{14}, F_7 / F_8, F_6 / F_9;
3) duality: F_1 / F_{14}, F_7 / F_8, F_6 / F_9;
4) partial complement
$F_1 / F_7, F_7 / F_1, F_8 / F_{14}, F_{14} / F_8$

In table 2 we present the Karnaugh diagram and the normal and inverse disjunctive and conjunctive forms for commutative LF, from which we can establish the equivalence, complement, duality and partial complement relations between the presented LF. These binary relations between different commutative LF from LA are presented more specifically in table 3. Analyzing binary relations between different elemental commutative LF from LA, determined on the same definition domain as argument tuples, allows us to point out the following types of binary relations between LF:

Table 1. Logic functions

x_1	x_2	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 2. Karnaugh diagrams and Normal forms for LF representatino

a \ bc	00	01	11	10
0	0	0	0	0
1	0	0	1	0

Disjunctive form: $F_1^D = a\cdot b\cdot c$, (1.1)
Inverse disjunctive form: $\overline{F}_1^D = \overline{a}\vee\overline{b}\vee\overline{c} = \overline{a\cdot b\cdot c}$, (1.2)
Conjunctive form: $F_1^C = (a)\cdot(b)\cdot(c)$, (1.3)
Inverse conjunctive form: $\overline{F}_1^C = (\overline{a})\vee(\overline{b})\vee(\overline{c})$, (1.4)

a \ bc	00	01	11	10
0	0	1	1	1
1	1	1	1	1

Disjunctive form: $F_2^D = a\vee b\vee c$, (2.1)
Inverse disjunctive form: $\overline{F}_2^D = \overline{a}\cdot\overline{b}\cdot\overline{c} = \overline{a\vee b\vee c}$, (2.2)
Conjunctive form: $F_2^C = (a)\vee(b)\vee(c)$, (2.3)
Inverse conjunctive form: $\overline{F}_2^C = \overline{a}\cdot\overline{b}\cdot\overline{c} = \overline{a\vee b\vee c}$, (2.4)

a \ bc	00	01	11	10
0	1	1	1	1
1	1	1	0	1

Disjunctive form: $F_3^D = \overline{a}\vee\overline{b}\vee\overline{c} = \overline{a\cdot b\cdot c}$, (3.1)
Inverse disjunctive form: $\overline{F}_3^D = a\cdot b\cdot c$, (3.2)
Conjunctive form: $F_3^C = \overline{a}\vee\overline{b}\vee\overline{c} = \overline{a\cdot b\cdot c}$, (3.3)
Inverse conjunctive form: $\overline{F}_3^C = a\cdot b\cdot c$, (3.4)

a \ bc	00	01	11	10
0	1	0	0	0
1	0	0	0	0

Disjunctive form: $F_4^D = \overline{a}\cdot\overline{b}\cdot\overline{c} = \overline{a\vee b\vee c}$, (4.1)
Inverse disjunctive form: $\overline{F}_4^D = a\vee b\vee c$, (4.2)
Conjunctive form: $F_4^C = (\overline{a})\cdot(\overline{b})\cdot(\overline{c}) = \overline{a\vee b\vee c}$, (4.3)
Inverse conjunctive form: $\overline{F}_4^C = a\vee b\vee c$, (4.4)

a \ bc	00	01	11	10
0	1	0	0	0
1	0	0	0	0

Disjunctive form: $F_5^D = \overline{a}\cdot\overline{b}\cdot\overline{c}$, (5.1)
Inverse disjunctive form: $\overline{F}_5^D = a\vee b\vee c$, (5.2)
Conjunctive form: $F_5^C = (\overline{a})\cdot(\overline{b})\cdot(\overline{c})$, (5.3)
Inverse conjunctive form: $\overline{F}_5^C = a\vee b\vee c$, (5.4)

a \ bc	00	01	11	10
0	1	1	1	1
1	1	1	0	1

Disjunctive form: $F_6^D = \overline{a}\vee\overline{b}\vee\overline{c}$, (6.1)
Inverse disjunctive form: $\overline{F}_6^D = a\cdot b\cdot c$, (6.2)
Conjunctive form: $F_6^C = \overline{a}\vee\overline{b}\vee\overline{c}$, (6.3)
Inverse conjunctive form: $\overline{F}_6^C = a\cdot b\cdot c$, (6.4)

a \ bc	00	01	11	10
0	0	1	1	1
1	1	1	1	1

Disjunctive form: $F_7^D = a\vee b\vee c$, (7.1)
Inverse disjunctive form: $\overline{F}_7^D = \overline{a}\cdot\overline{b}\cdot\overline{c} = \overline{a\vee b\vee c}$, (7.2)
Conjunctive form: $F_7^C = (a)\vee(b)\vee(c)$, (7.3)
Inverse conjunctive form: $\overline{F}_7^C = \overline{a}\cdot\overline{b}\cdot\overline{c} = \overline{a\vee b\vee c}$, (7.4)

a \ bc	00	01	11	10
0	0	0	0	0
1	0	0	1	0

Disjunctive form: $F_8^D = a\cdot b\cdot c$, (8.1)
Inverse disjunctive form: $\overline{F}_8^D = \overline{a}\vee\overline{b}\vee\overline{c} = \overline{a\cdot b\cdot c}$, (8.2)
Conjunctive form: $F_8^C = (a)\cdot(b)\cdot(c)$, (8.3)
Inverse conjunctive form: $\overline{F}_8^C = (\overline{a})\vee(\overline{b})\vee(\overline{c})$, (8.4)

a \ bc	00	01	11	10
0	0	1	0	1
1	1	0	1	0

Disjunctive form: $F_9^D = \overline{a}\cdot\overline{b}\cdot c\vee\overline{a}\cdot b\cdot\overline{c}\vee a\cdot\overline{b}\cdot\overline{c}\vee a\cdot b\cdot c$, (9.1)
Inverse disjunctive form: $\overline{F}_9^D = \overline{a}\cdot\overline{b}\cdot\overline{c}\vee\overline{a}\cdot b\cdot c\vee a\cdot\overline{b}\cdot c\vee a\cdot b\cdot\overline{c}$, (9.2)
Conjunctive form: $F_8^C = a\cdot b\cdot c\vee a\overline{b}\overline{c}\vee\overline{a}\cdot b\cdot\overline{c}\vee\overline{a}\cdot\overline{b}\cdot c$, (9.3)
Inverse conjunctive form: $\overline{F}_9^C = a\cdot b\cdot\overline{c}\vee a\cdot\overline{b}\cdot c\vee\overline{a}\cdot b\cdot c\vee\overline{a}\cdot\overline{b}\cdot\overline{c}$, (9.4)

a \ bc	00	01	11	10
0	1	0	1	0
1	0	1	0	1

Disjunctive form: $F_{10}^D = \overline{a}\cdot\overline{b}\cdot\overline{c}\vee\overline{a}\cdot b\cdot c\vee a\cdot\overline{b}\cdot c\vee a\cdot b\cdot\overline{c}$, (10.1)
Inverse disjunctive form: $\overline{F}_{10}^D = \overline{a}\cdot\overline{b}\cdot c\vee\overline{a}\cdot b\cdot\overline{c}\vee a\cdot\overline{b}\cdot\overline{c}\vee a\cdot b\cdot c$, (10.2)
Conjunctive form: $F_{10}^C = a\cdot b\cdot\overline{c}\vee\overline{a}\cdot\overline{b}\cdot\overline{c}\vee a\cdot\overline{b}\cdot c\vee\overline{a}\cdot b\cdot c$, (10.3)
Inverse conjunctive form: $\overline{F}_{10}^C = \overline{a}\cdot\overline{b}\cdot c\vee\overline{a}\cdot b\cdot\overline{c}\vee a\cdot\overline{b}\cdot\overline{c}\vee a\cdot b\cdot c$, (10.4)

Table 3. Binary relations between some logic algebra commutative functions

tuples	$F_{1norm}^{\vee}$	$F_{2norm}^{\wedge}$	F_1^C	F_2^C	F_1^D	F_2^D	$\overline{F}_1^D$	$\overline{F}_2^D$	F_3	F_3^C
	$a\vee b\vee c$	$a\cdot b\cdot c$	$\overline{a\vee b\vee c}$	$\overline{a\cdot b\cdot c}$	$\bar a\cdot\bar b\cdot\bar c$	$\bar a\vee\bar b\vee\bar c$	$\overline{\bar a\cdot\bar b\cdot\bar c}= a\vee b\vee c$	$\overline{\bar a\vee\bar b\vee\bar c}= a\cdot b\cdot c$	IMPARITY $a\oplus b\oplus c$	PARITY $a\overline{\oplus} b\overline{\oplus} c$
‹000›	0	0	1	1	1	1	0	0	0	1
‹001›	1	0	0	1	0	1	1	0	1	0
‹010›	1	0	0	1	0	1	1	0	1	0
‹011›	1	0	0	1	0	1	1	0	0	1
‹100›	1	0	0	1	0	1	1	0	1	0
‹101›	1	0	0	1	0	1	1	0	0	1
‹110›	1	0	0	1	0	1	1	0	0	1
‹111›	1	1	0	0	0	0	1	1	1	0

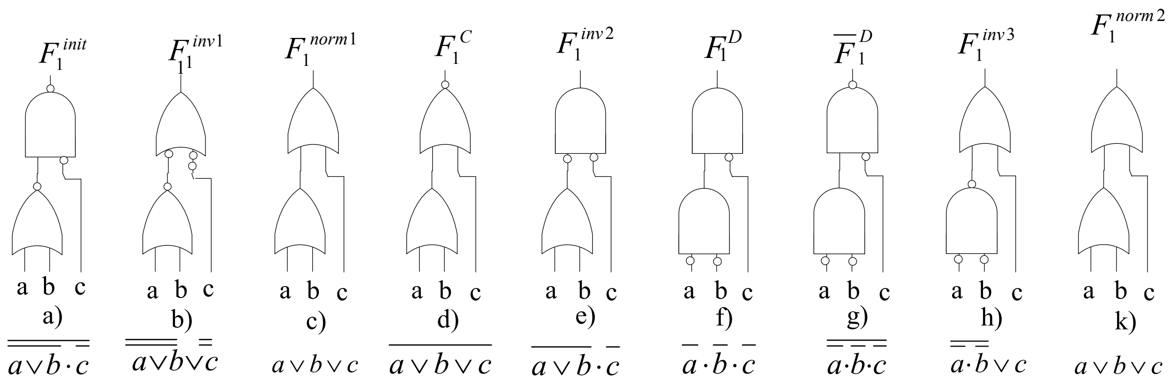

Fig. 1. LF modification and binary relations between LF

Table 4. Logic algebra commutative functions binary relations

tuples	F_1^{init}	F_1^{inv1}	F_1^{norm1}	F_1^C	F_1^{inv2}	F_1^D	$\overline{F}_1^D$	F_1^{inv3}	F_1^{norm2}	$F_2^{\wedge}$
$\langle X_K\rangle$	$\overline{\overline{a\vee b}\cdot\overline{\overline{c}}}$	$\overline{\overline{a\vee b}}\vee\overline{\overline{c}}$	$a\vee b\vee c$	$\overline{a\vee b\vee c}$	$\overline{a\vee b}\cdot\bar c$	$\bar a\cdot\bar b\cdot\bar c$	$\overline{\bar a\cdot\bar b\cdot\bar c}$	$\bar{\bar a}\cdot\bar{\bar b}\vee c$	$a\vee b\vee c$	$a\wedge b\wedge c$
‹000›	0	0	0	1	1	1	0	0	0	0
‹001›	1	1	1	0	0	0	1	1	1	0
‹010›	1	1	1	0	0	0	1	1	1	0
‹011›	1	1	1	0	0	0	1	1	1	0
‹100›	1	1	1	0	0	0	1	1	1	0
‹101›	1	1	1	0	0	0	1	1	1	0
‹110›	1	1	1	0	0	0	1	1	1	0
‹111›	1	1	1	0	0	0	1	1	1	1

1) equivalence: $F_{1norm}^{\vee}/\overline{F}_2^D$, $F_{2norm}^{\wedge}/\overline{F}_1^D$, F_2^C/F_2^D, F_1^C/F_1^D;

2) complement: $F_{1norm}^{\vee}/F_2^C$, $F_{2norm}^{\wedge}/F_1^C$, $F_2^D/\overline{F}_2^D$, $F_1^D/\overline{F}_1^D$;

3) duality: $F_{1norm}^{\vee}/F_2^D$, $F_{2norm}^{\wedge}/F_1^D$, $F_2^D/\overline{F}_2^D$, $F_1^D/\overline{F}_1^D$;

4) partial complement: $F_{1norm}^{\vee}/F_{2norm}^{\wedge}$, F_1^C/F_2^C, F_1^D/F_2^D, $\overline{F}_1^D/\overline{F}_2^D$

However considering the different modification steps (fig. 1) for a simple circuit (chosen especially for the additional proof of specific properties) allows a more profound understanding of the design problem and functional-logic relations. The analytical descriptions of the structure modifications in fig. 1 and the relations between different logic-functional representation forms is given in table 4, from where we can observe that equivalence, duality, negation and complement relations are characteristic for DC. Also, the duality, negation and complement are logically equivalent (but not structurally), which are characterized, respectively, by the presence of negation only of the primary variable, intermediary variables for different logic levels or only the exit variable (of the LF). In the paper the notion of duality is different from the standard (see [3, 4, 5, 6]).

On the other hand, from table 3 we can observe that only between certain *elemental functions* of LA, *partial complement relation* can additionally exist, giving to those relations a special significance. From table 3 we observe that the studied circuit represents a degenerate logic structure of type OR which is equivalent to the respective

OR logic gate (LG). The importance of this equivalence is in the fact that *for a logical gates verification tests are also diagnostic tests.*

The relations between LF represent a particular case, which requires special studies, consequent to the LF specific.

III. LF Special Relations

In order to establish the relations of a LF to another LF, satisfying DFT requirements, we will study the LF relations (tab. 3). Initially it is necessary to establish the relations between the values of the ordered couple $(X_k, F^{\vee}_{1norm})$ of each tuple and the respective logic value of the disjunctive (or conjunctive) normal function. Subsequently we will establish the relations between the values of each tuple X_k and the respective values for each LF in table 3. Afterwards binary relations can be established between the values of the ordered LF couples from table 3, determined on the same definition domain. It is clear that in all the cases $F^{\vee}_{1norm}$ serves as reference LF, comparing it to another LF of the couple allowing a certain type of relation.

The formalization this approach, also for non-trivial LF, is based on the following notions and definitions.

1. A binary relation *R* defined between the elements of the sets A and B is a subset of the Cartesian product $R \subseteq A \times B$.

2. If A = B, then the binary relation $R \subseteq A \times A$ on the set A is a subset of the Cartesian product, meaning a set of tuples (x,y) of elements from A. The fact that the tuple $(x, y) \in A$ is noted xRy or $(x, y) \in R$.

3. If A and B are two sets and by a procedure one element $x \in A$ correspond to one and only one $y \in B$

4. The set of the first – x elements of the tuple (x, y) is called the definition (starting) domain of the relation R and is written *Dom R*:

$$Dom \quad R = \{ x \mid \exists y ((x, y) \in R)\} \tag{2}$$

5. The set of all the secondary elements – y of the tuple (x,y) from R is called the value domain (arrival) of R relation and is written by *Im R*:

$$\mathrm{Im}\, R = \{y \mid \exists x((x,y) \in R)\} \tag{3}$$

6. The image F of the set A in set B is called the rule by which to each element $x \in A$ corresponds the element $y \in B$, which can be put down in the following way:

$$F : A \to B \quad or \quad y = F(x) \tag{4}$$

7. A reciprocal one-way relation is a binary relation R, defined on a certain set, which is different by the fact that for each value of *x* corresponds a unique value of *y* and for each value of *y* corresponds a unique value of *x*.

8. The LF studied in tab 4 represent sets which contain only two types of elements: 0 and 1.

9. Two LF are equivalent - $F_1^{norm\,1} \underline{\underline{E}}\, F_1^{init}$ if $F_1^{norm\,1}(X_k) = F_1^{init}(X_k)$ for any $k \in \overline{0 \div 2^n - 1}$.

10. Two LF are complements $F_1^{norm\,1} \underline{\underline{C}}\, F_1^{C}$, if $F_1^{norm\,1}(X_k) = \overline{F}_1^{C}(X_k)$, for any $k \in \overline{0 \div 2^n - 1}$.

11. Two LF are dual - $F_1^{norm\,1} \underline{\underline{D}}\, F_1^{D}$, if $F_1^{norm\,1}(X_k) = \overline{F}_1^{D}(X_k)$, for any $k \in \overline{0 \div 2^n - 1}$.

12. Two LF are partial complements - $F_1^{norm\,2} \underline{\underline{PC}}\, F_2^{\wedge}$, if $F_1^{norm\,2}(X_k) = \overline{F}_2^{\wedge}(X_k)$, for any $k \in \overline{1 \div 2^n - 2}$ and if $F_1^{norm\,2}(X_k) = F_2^{\wedge}(X_k)$, for $k = 0$ & $k = 2^n - 1$

IV. Conclusions

This paper represents a study in relational properties of LF of LA and also the acquisition in premiere of the mathematical description of the relational properties of LA functions. The original results of the study have an essential importance in solving the DFT problem and represent indispensable elements for the concepts of elaborating digital structures, for the formalization methods of the DFT process and for the respective know-how. This study represents an important step in creating and formalizing the new concepts for designing testable structures destined for improving the technological efficiency of the present integrated circuits manufacturing processes. Some of the conclusions of the study follow:

1. The necessary and sufficient condition for the existence of binary relations between LF is the coincidence of *Dom R* of these LF.

2. LF binary relations represent an important problem, for the moment not sufficiently studied.

3. Duality, negation and complement relations are logically equivalent and actually represent relations between the initial functions and each of the three different aspects of the inverse logic function. The importance of each relation is determined by the individual role it plays in the design process.

4. Partial complement binary relation cannot be obtained for composed functions, irrespective of the modifications done to the LF (see last column in table 4). This relation appears only between elemental logic functions, a fact which proves its importance and specificity in creating elemental digital structures as a base for the new DFT paradigm.

REFERENCES

[1] Ion Cojocaru, Study in Boolean algebra functions binary relations. – Ongoing publication

[2] K. Kinosita, K. Asada, O. Karatsu, Logical design of VLSI. – Loghiceskoe proiectirovanie SBIS – Iwanami Shoten Publishers, Tokyo, 1985, 304 pag. Trad. în limba rusă, 1988.

[3] Wakerly J. F., Circuite digitale. Principiile şi practicile folosite în proiectare, Bucureşti: Teora, – 2002. – 928 p.

[4] M. Rafiquzzaman, Fundamentals of Digital Logic and Microcomputer Design.- Fifth Edition, California, WILEY INTERSCIENCE – John Wiley&Sons inc., 2005, 840 pag

[5] Wilkinson B., Digital System Design. – 4nd edition, Prentice Hall, London, 2004, 320 p.

[6] Holdsworth B., Digital Logic Design. – Butterworth & Co (Publishers), LTD, London, 1982, 285 p.

[7] Белоусов А.И., Ткачев С.Б., Дискретная математика.- Изд-во МГТУ им. Н.Э. Баумана, 2004 г., -744 с