The BRDF Models Based on the Cosine-quadratic Function

Alexander ROMAHYUK, Yuri LYASHENKO
Vinnitsa National Technical University
str. Khmelnitske Shosse, 95, 21021, Vinnitsa, Ukraine
ran12345@mail.ru, Yuri.Lyashenko@gmail.com

***Abstract* — This paper describes several approaches to approximate bidirectional reflectance distribution function (BRDF) by cosine-quadratic functions. Since BRDF is the most computationally complicated part of calculating the color intensity according to the Phong illumination model it needs to be simplified. Herein, several approximations are provided. The advantages of them are numerous: easy hardware implementation, less relative error, than most widespread approximations have, fast to compute.**

***Index Terms* — Approximation methods, computer graphics, Color graphics, Graphics, Rendering, BRDF**

I. INTRODUCTION

The most complicated and resource-intensive computing at the rendering stage takes place in the shading of 3-D objects having voluminosity effect.

The color intensity and coordinates are detected for every pixel on the surface in the process of shading. Taking into consideration, that high resolution displays are used for representing realistic images, shading process takes a lot of time. Especially it reveals when illumination models representing specular constituent of the color are used. That's why the question of the increasing shading production in computer graphic systems is very urgent one.

II. EXISTING APPROACHES AND PROBLEM DEFINITION

The intensity of pixels' color according to the Phong method is detected using following:

$$I = I_a k_a + I_l (k_d \cos\psi + k_s \cos^n \lambda),$$

where I_a, I_l - intensities of sparse and directed light sources correspondingly, k_a, k_d, k_s - sparse, diffusive and reflecting light coefficients, ψ - angle between the direction of light and normal vector, λ - angle between the direction of reflected light and the observer, n - surface brightness coefficient, $\cos^n \lambda$ - BRDF, represents surface optical properties.

In shading, the most resource-intensive procedure is the computing $\cos^n \gamma$, used in Phong and Blin illumination models [1]. K. Schlick [2] proposed approximation of the function $\cos^n \gamma$, that is used for computing intensity of the mirror constituent of the color in Phong and Blin illumination models, with function $\cos\gamma / (n - n\cos\gamma + \cos\gamma)$. Analysis showed this approach to provide satisfactory accuracy only within the highlight epicenter representation. Beyond this area discrepancy with results, received according to the Blin illumination model is observed.

Function $\cos^n \gamma$ is expanded into Taylor series in method [3], proposed by R.F.Lion. Instead of the angle λ between reflected light and observer, the length of chord between mentioned vectors is used: $\gamma = |\vec{D}| = |\vec{N} - \vec{H}|$.

The expression $1/\sqrt{\vec{N} \cdot \vec{N}}$, that is used for normalizing of normal vector $\vec{N}$, is also expanded into Taylor series. Only first 3 terms are used. This allows elimination from the computing process division and square root operations. Replacement of the angle with chord length doesn't influence greatly upon the accuracy of the calculations, but only for low values of the angles, and using limited quantity of Taylor series doesn't allow normal vectors normalizing, therefore approximation ain't accurate enough.

These and other approaches have in common two major disadvantages: computation and hardware implementation complexity. The challenge is to provide BRDF model without mentioned issues.

III. COSINE-QUADRATIC FUNCTION APPROACH

The idea is to approximate BRDF $cos^n \gamma$ with function $W(n, \gamma) = \left(\zeta \cdot \left(cos\,\gamma - 1\right) + 1\right)^2$ on conditions, that $0 \le \gamma \le \pi/2$. This function has been chosen because of following:

a) the generatrix function for both of them is a cosine function;

b) when $\gamma = 0$ $cos^n \gamma = \left(\zeta \cdot \left(cos\,\gamma - 1\right) + 1\right)^2 = 1,$ that satisfies boundary conditions;

c) both functions are positive in the range $0 \le \gamma \le \pi/2$;

d) function $\left(\zeta \cdot \left(cos\,\gamma - 1\right) + 1\right)^2$ reaches zero level, what is pre-condition for blooming zone formation;

e) the ζ coefficient provides a possibility to change the highlight area.

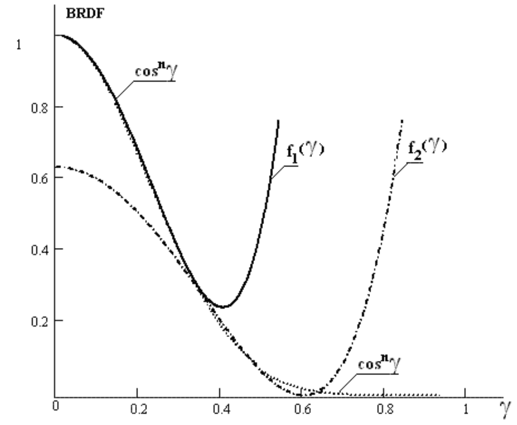

Figure 1. Graphics of $cos^n\gamma,\ \ f_1(\gamma),\ \ f_2(\gamma)$.

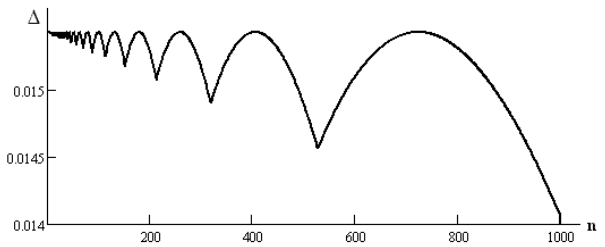

Figure 2. The graphics of the absolute error of the BRDF approximation by function $f_1(\gamma)$.

There're several ways to determine coefficient ζ in the expression $(\zeta\cdot(cos\gamma-1)+1)^2$. The first of them is provided below. Let's expand function $cos^n\gamma$ into the Taylor series and take first two terms

$cos^n\gamma\approx 1-n\cdot\gamma^2/2.$

The first two terms of the function $W(n,\gamma)$ Taylor series expansion follow:

$(\zeta\cdot(cos\gamma-1)+1)^2\approx 1-\zeta\cdot n\cdot x^2.$

Having equated right terms of both expressions the coefficient ζ is found, and $\zeta=\frac{1}{2}$. Thus,

$$cos^n\gamma\approx\left(\frac{n}{2}(cos\gamma-1)+1\right)^2.$$

Let's analyze the obtained BRDF.

The graphics of function $cos^n\gamma,\left(\frac{n}{2}(cos\gamma-1)+1\right)^2$ and of the Schlick function for $n=100$ are provided on the fig.3. The figure shows that suggested BRDF sufficiently approximates the highlight epicenter and monotonically decreases after the cusp. This satisfies the strategy of the specular color constituent formation. The fig.4 presents the graphics of the maximal relative errors of the approximation the $cos^n\gamma$ by the function $W(n,\gamma)$ for the highlight epicenter. The graphics show that in comparison to Schlick BRDF the accuracy of the highlight epicenter approximation has increased.

The function $W(n,\gamma)$ possesses a zero value when $cos(\gamma)=(n-2)/n$. And it goes without saying, this is a minimal value of the function, 'cause the square of a real term is never less than 0.

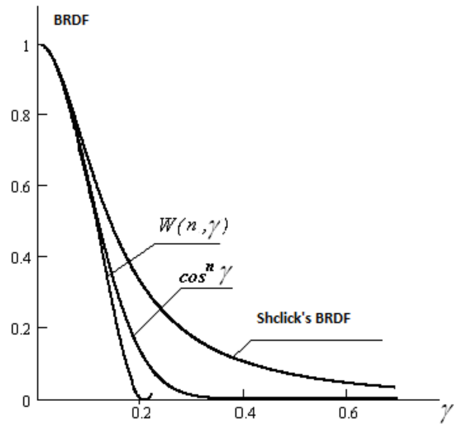

Figure 3. The BRDF Graphics.

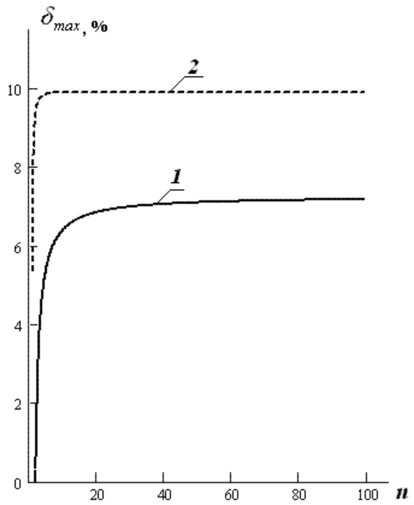

Figure 4. The graphics of the maximal relative errors for the highlight epicenter when BRDF is approximated by:

1– function $W(n,\gamma)$; 2 – Schlick's BRDF.

Let's prove that function $W(n,\gamma)$ monotonically decreases in the range $0, arcos((n-2)/n)$. The derivative of the $W(n,\gamma)$ follows

$$-(\frac{n}{2}(cos\gamma-1)+1)\cdot n\cdot sin\gamma.$$

In the range from 0 to π term $n\cdot sin\gamma$ is positive, therefore it doesn't influence upon the sign of the expression. The inequality $-(\frac{n}{2}(cos\gamma-1)+1)\le 0$ is correct for all $0\le\gamma\le arcos\frac{n-2}{n}$. This fits the variation interval from 1 to the minimal value.

The sufficient condition of function monotone decreasing is for its derivation to be negative within the range, what just has been proved.

Let's compare the obtained function with the Shclicks function [2], which has gained wide appliance in the software for 3D Graphics.

The graphics on the fig.4 show that in comparison to Schlick BRDF the accuracy of the highlight epicenter approximation has increased.

The relative error value at the cusp doesn't exceed 7,2 % for the $cos^n\gamma$ function, and almost equals 10% for the Shclick function.

To calculate the Schlicks BRDF it's necessary to perform multiplication, dividing and decrementing operations. The suggested function usage eliminates the laborious dividing operation from the computation process. It's necessary to compute only 2 multiplication operations, shift, increment and decrement. The last 3 operations are easily implemented on the assembly level. At the software implementation the time of determining the BRDF $W(n, \gamma)$ is less in 2,3 times.

This draws a conclusion that computation complexity of the function $W(x,n)$ is considerably lesser than Schlicks BRDF.

The hardware implementation of the function $W(n, \gamma)$ is simple, whereas usage of the dividing operation by the Schlicks BRDF complicates its hardware implementation.

Let's go through the peculiarities of the blooming zone formation.

Let's assume the threshold value is 2^{-q}, where q is selected according to the required specular color constituent accuracy. Then $cos^n\gamma$ has to be calculated within the following range – $12^{-q} \le cos^n\gamma \le 1$. Hence, $0 \le \gamma \le arccos\left(2^{\left(-\frac{q}{n}\right)}\right)$.

The following inequality is correct for the Schlick's BRDF

$$0 \le \frac{cos^n\gamma}{n - n\cdot cos^n\gamma + cos^n\gamma} \le 2^{-q}.$$

The worthwhile computation range for the Schlick's BRDF follows $0 \le \gamma \le arccos\frac{n}{2^q + n - 1}$.

Let's determine the variation range for the suggested BRDF. From the inequality

$0 \le \left(\frac{n}{2}(cos\gamma - 1) + 1\right)^2 \le 2^{-q}$ comes following

$$0 \le \gamma \le 1 - \frac{2}{n} + \frac{2}{n\cdot 2^{\frac{q}{2}}}.$$

Since $1 - \frac{2}{n} + \frac{2}{n\cdot 2^{\frac{q}{2}}} \le 2^{\left(-\frac{q}{n}\right)} \le \frac{n}{2^q + n - 1}$, Schlicks curve lies above $cos^n\gamma$, and curve $\left(\frac{n}{2}(cos\gamma - 1) + 1\right)^2$ - below (see fig.3).

The grave disadvantage of the Schlick's BRDF is that BRDF fades to zero level very slowly. (see fig.3), that leads to unnatural illumination of the graphic object and requires superfluous computations the argument variation range is increased.

Let's find the ratio $\Re$, which shows in how many times the arguments of functions $cos^n\gamma$ and suggested BRDF differ, when they reach the 2^{-q} level. Let's compare it with the Schlick's case too.

It's clear, that $\Re$ determines the correlation of the highlight for various BRDFs.

For the functions $cos^n\gamma$ and $W(x,n)$ $\Re = \frac{arccos\left(2^{\left(-\frac{q}{n}\right)}\right)}{1 - \frac{2}{n} + \frac{2}{n\cdot 2^{\frac{q}{2}}}}$. For the function $cos^n\gamma$ and Schlicks one $\Re = \frac{arccos\left(2^{\left(-\frac{q}{n}\right)}\right)}{a\,arccos\frac{n}{2^q + n - 1}}$.

The graphics on the fig.5 show that in case of appliance the suggested BRDF blooming is calculated within considerably lesser range in comparison to the Schlicks case.

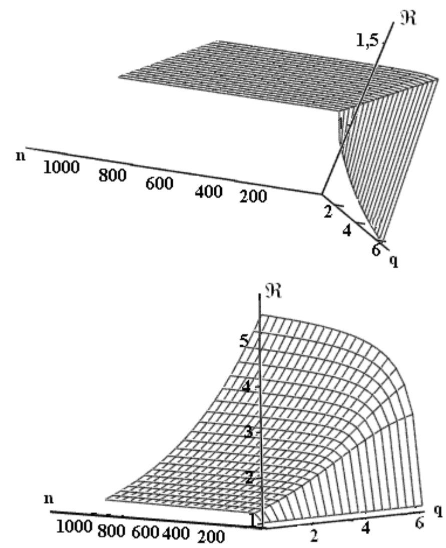

Figure 5. The variation graphics of $\Re$ for :

a) $cos^n\gamma$ and $W(n, \gamma)$; b) $cos^n\gamma$ and Schlick's BRDF.

The important point in usage of the suggested BDF is a nullification of its values after it once has reached the zero level. This can be easily hardware implemented. The $cos\gamma$ value is easy obtained through the inner product of the normalized vectors. For each n in memory block we have to store boundary value of the cosine, starting from which, the further computation of the BRDF is to be ceased.

The radius of the curvature is an important parameter for the blooming formation [4, 5]. It determines the outline sharpness of the highlight. The second derivative of W follows

$$W''_x = \frac{1}{2}\cdot n^2\cdot sin^2\gamma - n\cdot cos\gamma\cdot\left(\frac{1}{2}\cdot n\cdot(cos\gamma - 1) + 1\right).$$

The curvature radius is

$$r=\frac{\left(1+(W_x')^2\right)^{\frac{3}{2}}}{|W_x''|}=\frac{\left(1+(-(\frac{n}{2}(\cos\gamma-1)+1)\cdot n\cdot\sin\gamma)^2\right)^{\frac{3}{2}}}{\left|\frac{1}{2}\cdot n^2\cdot\sin^2\gamma-n\cdot\cos\gamma\cdot(\frac{1}{2}\cdot n\cdot(\cos\gamma-1)+1)\right|}.$$

Considering γ has value $ar\cos\frac{n-2}{n}$ it's easy to obtain $r=1/2\cdot n-2$. The last expression shows that r decreases, when n increases, what is typical for the classic BRDF.

There's another approach to determine ζ.

The curve $\left(\zeta\cdot(\cos\gamma-1)+1\right)^2$ reaches the zero value at the point $\zeta=a\cos((\zeta-1)/\zeta)$. For the suggested BRDF we have $\cos^n\gamma=\left(\zeta\cdot(\cos\gamma-1)+1\right)^2$.

Let's multiply both sides of the expression to $\sin\gamma$ and integrate both sides. The integration range is from the zero value of the argument to the point, where functions possess zero level.

From the equation $\left(\zeta\cdot(\cos\gamma-1)+1\right)^2=0$ comes, that $\gamma=ar\cos((\zeta-1)/\zeta)$. The define integral is

$$\int_0^{ar\cos\left(\frac{\zeta-1}{\zeta}\right)}\left[\zeta\cdot(\cos\gamma-1)+1\right]^2\cdot\sin\gamma d\gamma=\frac{1}{3\cdot\zeta}.$$

Let's do the same for $\cos^n\gamma$

$$\int_0^{\pi/2}\cos^n\gamma\cdot\sin\gamma\cdot d\gamma=\frac{\cos^{n+1}\gamma}{n+1}\Big|_0^{\pi/2}=\frac{1}{n+1}.$$

Having compared right sides of the both equations we obtain $\zeta=(n+1)/3$.

Thus, a new BRDF is $W_2=\left(\frac{(n+1)}{3})\cdot(\cos\gamma-1)+1\right)^2$.

The typical for this function is even distribution of the absolute error within the whole computation interval. The fig. 6 contains the example of BRDF, when $n=10$, and graphic of the dependence of the maximal relative error on the specular coefficient. The fig.7 shows the example of the 3D object created by using the W_2 function.

To determine the max relative error let's find the derivative of the expression $\left[\left(\frac{(n+1)}{3}\cdot(\cos\gamma-1)+1\right)^2-\cos^n\gamma\right]/\cos^n\gamma$ and equal it to zero. The maximal error takes place at the point $\gamma=arctg\left((1+2\cdot n)^{\frac{1}{2}}/n\right)$.

Maximal absolute error doesn't exceed the value 0,1.

Another approach of determining ζ is to equal values of functions $\cos^n\gamma$, $\left(\zeta\cdot(\cos\gamma-1)+1\right)^2$ at the certain point t. Since $\cos(t)^n=Q$, then $t=e^{\frac{\ln(Q)}{n}}$.

The following equation is obtained $Q=\left(\zeta\cdot\left(\cos e^{\frac{\ln(Q)}{n}}-1\right)+1\right)^2$.

The solution is: $\zeta_1=\frac{\sqrt{Q}-1}{e^{\frac{\ln(Q)}{n}}-1}$, $\zeta_2=\frac{-\sqrt{Q}-1}{e^{\frac{\ln(Q)}{n}}-1}$.

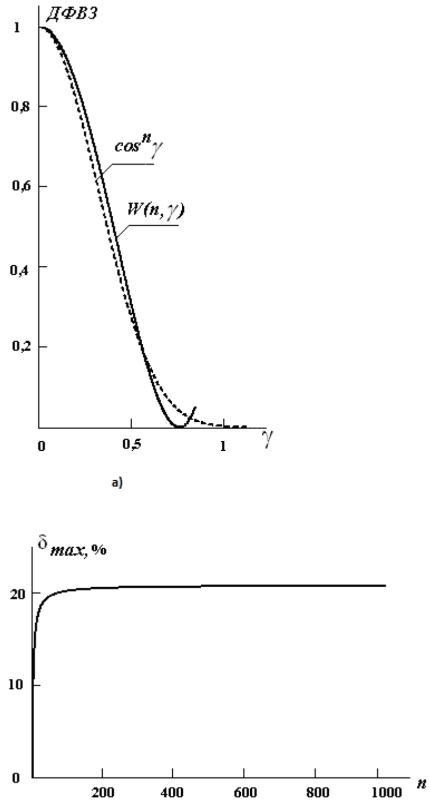

Figure 6. Graphics of:

a) BRDF $W_1(n,\gamma)$ for n=10;

b) the dependence of max relative error on n

Figure 7. The example of formation the 3D object with usage of the suggested BRDF.

The solution ζ_2 is inacceptable, because the function will have a common point when monotonically increase. Thus, the new BRDF model is

$$W_3=\left(\frac{\sqrt{Q}-1}{e^{\frac{\ln(Q)}{n}}-1}\cdot(\cos\gamma-1)+1\right)^2.$$

Changing Q one can change the shape of the curve and the accuracy of the highlight formation. The investigations revealed that when Q has value 0,5 the accuracy is the best. In this case

$$W_3 \approx \left(\frac{7 \cdot n}{16} \cdot (\cos\gamma - 1) + 1 \right)^2 .$$

For the certain n the max relative error at the highlight epicenter formation occurs at the point

$$\gamma = arctg\left((2 \cdot \sqrt{7 \cdot n - 15}\,) / (7 \cdot n - 16\,) \right).$$

Down to the level 0,45 the max absolute error for all $n \geq 20$ doesn't exceed 0,015. For another values of n this value is less than 0,02. For the Schlick's function even down to the cusp level max absolute error reaches 0,06, what is in 4 times greater.

The curve reaches the zero level at the point $\gamma = arc\cos(1 - 16/7 \cdot n)$. The curvature radius is $r = \frac{1}{\frac{7}{4} \cdot n - 2}$ and it is greater than for the curve W_1.

The value $\frac{7 \cdot n}{16}$ can easily be obtained on the assembly level. It's enough to subtract n from $8 \cdot n$ and shift the obtained value to 4 bits. To determine W_3 it's enough to perform 2 multiplication operations, 1 decrement and increment, subtraction and 7 shifts. When use the Pentium M processor, the computation time is decreased in 1,69 times in comparison with the Schlick's BRDF.

IV. CONCLUSIONS

In this article some BRDFs, having simple hardware and software implementation are provided.

Function $(n/2 \cdot (\cos\gamma - 1) + 1)^2$ needs only multiplication, subtracting and decrementing operations. The suggested function usage eliminates the laborious dividing operation from the computation process. It's necessary to compute only 2 multiplication operations, shift, increment and decrement. The last 3 operations are easily implemented on the assembly level.

At the software implementation the time of determining this BRDF is less in 2,3 times in comparison with Schlick BRDF.

Function $W_3 \approx \left(\frac{7 \cdot n}{16} \cdot (\cos\gamma - 1) + 1 \right)^2$ needs enough to perform 2 multiplication operations, 1 decrement and increment, subtraction and 7 shifts.

When use the Pentium M processor, the computation time is decreased in 1,69 times in comparison with the Schlick's BRDF.

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