Optimization of an Improved Nyquist Filter With Piece-Wise Polynomial Frequency Characteristic

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Abstract—In this paper, we optimize an improved Nyquist pulse generated by CC3 filter with piece-wise parabolic frequency characteristic. Its asymptotic decay rate is $t^{-2}$. The Nyquist pulse optimized here has a closed-form expression in time domain. Its performance is superior to previously reported pulses when sampled with a time offset. The optimization of the pulse was performed using Nelder-Mead method in order to minimize the symbol error probability. We have considered a timing offset that takes values between 0 and 0.3 and have determined the optimal values of the parameter value $b$ as a function of the excess bandwidth $a$, given an imposed value of the parameter $m$ that results from practical reasons.

Keywords— Inter-symbol interference, Nyquist filter, error probability, filter optimization

I. INTRODUCTION

In order to communicate efficiently, the frequency spectrum of the transmitted/received signal has to be bandwidth limited. The larger the bandwidth the stronger the noise power is. An ideal low-pass filter is not realizable and its impulse response described by a $\sin x / x$ function is not indicated, as with a small synchronization error the sampling will take place with a rather small random shift with respect to the ideal time instants, which results in non-zero inter-symbol interference. Moreover, the introduced errors decrease in time as $t^{-1}$, which corresponds to a time series $1 / n$, which is not a convergent one. This means that the pulse tails can combine in an additive manner and introduce difficulties in interpreting the binary value associated to the pulse, their effect being known as ISI.

The real sampling time instants differ from the ideal ones, as they are generated by a synchronization circuit that extracts the clock information from the received signal that was distorted by the communication channel and is accompanied by noise. This introduces a parasitic phase modulation known as jitter and the real sampling instants are distributed around the ideal ones.

In a practical receiver the information is recovered according to Nyquist’s first criterion sampling the time response of a Nyquist filter in real conditions. The sampling clock signal is provided by a synchronization circuit and is corrupted by noise, phase distortion and tracking error. In mobile communications one should also consider multipath and Doppler shift. That is the sampling is performed with a timing error and non-zero inter-symbol interference. As a result an increase of the error probability takes place and the quality of transmission may diminish.

The time response described by a $\sin c$ function is not realizable, as it implies infinite delay. The research aimed at finding a most suitable Nyquist filter was centered on the following goals:

- to be produced by a filter that is realizable physically and provides smooth attenuation;
- to be robust to synchronization errors, when sampling takes place near ideal time instants;
- to preserve the zero crossings of the $\sin c$ - type response.

Initially the researchers worked in time domain, searching for pulses that have the continuity of frequency characteristic and of a number of its derivatives at the right end of the definition interval, where all should be equal to zero. The continuity of the frequency response and of its $k-1$ derivatives ensures a fast damping of the time response according to the law $t^{-(k+1)}$ and the reduction of the energy stored in the tails of $h(t)$ and by consequence the ISI reduction.

A good pulse was found to be the raised cosine (RC) one, which is composed of a flat zone followed by one with a sinusoidal decay. The RC filter became a de facto standard in telecommunications and is most popular. It possesses odd-symmetry with respect to the ideal cut-off frequency and is described in frequency domain by:

$$H_{RC}(f) = \begin{cases} 1, & |f/B| \leq 1 - \alpha \\ 1 + \cos \left( \pi \left| \frac{1 - B(1 - \alpha)}{2B\alpha} \right| \right), & 1 - \alpha \leq |f/B| \leq 1 + \alpha \\ 0, & |f/B| > 1 + \alpha \end{cases}$$

(1)
Fig. 1. Raised cosine filter characteristics

Here B represents the bandwidth of the brick-wall filter and \( \alpha \) is the roll-off factor or excess bandwidth. The frequency response of the RC filter is represented in Fig. 1 for three values of \( \alpha \). The scaled time function is given by:

\[
h_{RC}(t) = \frac{\sin(\pi t / T)}{\pi t / T} \cdot \frac{\cos(\pi \alpha t / T)}{1 - 4\alpha^2 t^2 / T^2}
\]

and decays as \( t^{-3} \). This is due to the fact that its frequency characteristic and its first two derivatives possess continuity throughout the definition domain and are null at \( B(l + \alpha) \).

II. IMPROVED NYQUIST FILTERS

A new perspective and line of research was introduced by Beaulieu and his co-workers in 2001 [1]. In short, this is known as improved Nyquist filters (INFs). The new Nyquist pulse introduced by Beaulieu e.a. performed better than the RC pulse when the data are recovered by sampling the time response at the receiver’s site in real conditions, that is with a timing error given by the synchronization circuit in the same conditions (assuming the same values of signal-to-noise ratio and excess bandwidth) [1].

The frequency spectrum of BTRC pulse is defined as

\[
H_{FE}(f) = \begin{cases} 
1, & |f| / B \leq 1 - \alpha \\
\frac{1}{e^{|\beta|(|1|+\alpha)-1}}, & 1 - \alpha \leq |f| / B < 1 \\
1 - \frac{1}{e^{|\beta|(|1|+\alpha)-1}}, & 1 / B < |f| / B \leq 1 + \alpha \\
0, & B(1 + \alpha) < |f| / B 
\end{cases}
\]

where: \( \beta = (\ln 2) / (\alpha B) \).

The pulse proposed and investigated by Beaulieu e.a. is known under different names, such as Parametric Exponential Pulse (n=1) [3] or Better Than Raised-Cosine (BTRC) [1]. Other researchers referred to this pulse as flipped-exponential (fexp or FE) [2], [11].

Fig. 2 illustrates the transfer functions of a Nyquist filter with raised cosine characteristic together with those of an improved Nyquist filter (INF) with flipped exponential characteristic. Three usual values of \( \alpha \), were considered.

The BTRC filter has a concave aspect of the transfer characteristic in the transition region beneath B, which determined a reduction of the magnitude of the first sidelobe and as a consequence a decrease of the symbol error rate when the sampling of the impulse response takes place with a timing error.

The concave aspect of the frequency characteristic in the range \( 1 - \alpha \leq |f| / B \leq 1 \) determines a convex aspect in the frequency interval \( 1 \leq |f| / B \leq 1 + \alpha \). In view of the odd-symmetry around B, a transfer of energy towards higher frequencies occurs and together with it a trade-off between the size of the largest sidelobe and the amplitudes of the remaining sidelobes that are increased at the expense of the magnitude of the first one.

The discontinuity of the first derivative of the frequency response at \( B(1 - \alpha) \) determines an asymptotic decay of the pulse that obeys the law \( t^{-2} \). This is compensated by a more open eye diagram that results in smaller bit error rate as a consequence of decreased sensitivity to timing jitter, as compared with the RC pulse.

Other pulses that outperform the BTRC pulse were reported later in [2-7]-[9-12].

III. CC3 FILTER

The CC3 filter was proposed and investigated in [11] along with three other families of INFs. They are based on a piece-wise frequency characteristic containing four parabolic pieces \( H_i(f) \) delimited by five border points, as shown in .Fig. 3.
for positive frequencies. Its time response will decay as $r^{-2}$ due to the presence of a discontinuity of the first derivative at $B(1-\alpha)$, and by consequence of the odd-symmetry also at $B(1+\alpha)$ [3].

In the sequel we will present an optimization of the CC3 filter [11].

\[
S(f) = \begin{cases} 
1, & \left| \frac{f}{B} \right| < 1 - \alpha \\
S_1(f), & 1 - \alpha < \left| \frac{f}{B} \right| < 1 - m \\
S_2(f), & 1 - m < \left| \frac{f}{B} \right| < 1 \\
S_3(f), & 1 < \left| \frac{f}{B} \right| < 1 - m \\
S_4(f), & 1 + m < \left| \frac{f}{B} \right| < 1 + \alpha \\
0, & \left| \frac{f}{B} \right| > 1 + \alpha 
\end{cases}
\]

Assuming $B = 1$,

\[
S_1(f) = b + \frac{(f - 1 + m)^2}{(\alpha - m)^2}(1 - b) \\
S_2(f) = b + \frac{(f - 1 + m)^2}{m^2}(1 - b) \\
S_3(f) = 1 - b + \frac{(f - 1 - m)^2}{m^2}(b - \frac{1}{2}) \\
S_4(f) = 1 - b + \frac{(f - 1 - m)^2}{(\alpha - m)^2}(b - 1)
\]

The impulse response results a sum of five individual contributions $s_i(t)$ brought by the chunks that compose the frequency characteristic. These are transposed into the time domain through Fourier transform:

\[
s_i(t) = \int_{N}^{M} S_i(f) \cos(2\pi f t) df
\]

where $M$ and $N$ are the delimiting points of a component piece and

\[
S_0(f) = 1
\]

Its corresponding impulse response results after several mathematical simplifications as

\[
s_{CC3}(t) = \frac{-2\sin(\pi t)}{(\alpha - m)^2 \alpha^2 \pi^2} \left( 2m^2(b - 1)\cos(\pi \alpha t) + d\cos(\pi m t) - (\alpha - m) ((2b - 1) (\alpha - m) - 2(b - 1) m^2 \pi t \sin(\pi \alpha t)) \right)
\]

where:

\[
d = \alpha^2 (2b - 1) + 2\alpha m(1 - 2b) + m^2
\]

The error probability $P_e$ follows the approach in [8] using Fourier series.

\[
P_e = \frac{1}{2} \sum_{M \text{ odd}}^{M} \left( e^{-\omega^2 \pi^2/2} \sin(\frac{m \omega h_0}{m}) \prod_{k=N_1}^{N_2} \cos(\frac{m \omega h_k}{m}) \right)
\]

where: $\omega = 2\pi/T_f$ is the angular frequency, $M$ is the number of coefficients considered in the truncation of the Fourier series of noise complementary distribution function; $T_f$ is the period used in the Fourier series; $p(t)$ is the pulse shape used, $T$ is the bit interval, $N_1$ and $N_2$ represent the number of interfering symbols before and after the transmitted symbol; and $h_k = p(t - kT)$ [8]. The calculations use $T_f = 40$ and $M = 61$ for $2^9$ interfering data pulses and $\text{SNR} = 15dB$.

Table I presents comparative results regarding the error probabilities for the CC3 pulse, poly pulse [4] and POWER pulse [7], considering binary antipodal signaling affected by timing errors. The superiority of the optimized CC3 filter specified by the found values for the parameters $m$ and $b$ is clearly evidenced.
The optimized values \((m, b)\) of the parameters that define the CC3 pulse are \((0.24, 0.67)\), \((0.24, 0.71)\) and \((0.24, 0.68)\) for \(\alpha = 0.25\) and \(t/T\) equal to 0.05, 0.1, and 0.2, respectively. For \(\alpha = 0.35\), the optimized values \((m, b)\) were determined as \((0.34, 0.71)\), \((0.34, 0.76)\), and \((0.34, 0.79)\), respectively. Finally, for \(\alpha = 0.5\) they were found as \(m = 0.49\), and \(b\) equal to 0.76, 0.79 and 0.84 for \(t/T\) equal to 0.05, 0.1, and 0.2, respectively.

IV. OPTIMIZATION OF THE CC3 FILTER

The optimization of the pulse was done based on Nelder-Mead optimization method, which is numerical and computationally compact. We aimed at minimizing the error probability for timing offset values between 0 and 0.3. We have searched for the optimal values of \(b\) as a function of the excess bandwidth, \(\alpha\) for an imposed value of the parameter \(m\) equal to \(\alpha - 0.01\) that resulted from practical reasons.

The steep decrease of the frequency characteristic near \(\beta(1-\alpha)\), produced by values of \(m\) close to \(\alpha\), determines spectral regrowth, i.e. an increase of the spectral lobes level. The optimal value of the free parameter \(b\) was found to obey a cubic relationship as a function of the normalized timing offset \(t_n = t/T\).

\[
b(t_n) = a_3 t_n^3 + a_2 t_n^2 + a_1 t_n + a_0
\]

The results are depicted in Fig. 4 for \(t_n = t/T\) varying between 0.01 and 0.3 and tabulated in Table II.

V. CONCLUSION

The bit error performance of an improved Nyquist filter denoted as CC3 was optimized for operation with timing error that produces inter-symbol interference using the Nelder-Mead technique. The free parameter \(b\) of the CC3 filter was found to obey a cubic relationship as a function of the normalized time offset \(t_n = t/T\).

### Table I. ISI ERROR PROBABILITY OF THE PROPOSED NYQUIST PULSES FOR \(N=2^N\) INTERFERING SYMBOLS AND SNR = 15dB

<table>
<thead>
<tr>
<th>(P_e)</th>
<th>(t/T=0.05)</th>
<th>(t/T=0.1)</th>
<th>(t/T=0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>poly((40,-100,85))</td>
<td>4.734e-8</td>
<td>8.834e-7</td>
<td>2.241e-4</td>
</tr>
<tr>
<td>POWER ((b=0.25))</td>
<td>4.576e-8</td>
<td>8.243e-7</td>
<td>2.048e-4</td>
</tr>
<tr>
<td>CC3</td>
<td>4.548e-8</td>
<td>8.189e-7</td>
<td>1.999e-4</td>
</tr>
<tr>
<td>poly((31,-80,69))</td>
<td>3.290e-8</td>
<td>3.839e-7</td>
<td>6.563e-5</td>
</tr>
<tr>
<td>POWER ((b=0.33))</td>
<td>3.103e-8</td>
<td>3.564e-7</td>
<td>6.434e-5</td>
</tr>
<tr>
<td>CC3</td>
<td>3.049e-8</td>
<td>3.490e-7</td>
<td>5.99e-5</td>
</tr>
<tr>
<td>poly((25,-64,55))</td>
<td>2.057e-8</td>
<td>1.354e-7</td>
<td>1.520e-5</td>
</tr>
<tr>
<td>POWER ((b=0.37))</td>
<td>1.945e-8</td>
<td>1.250e-7</td>
<td>1.662e-5</td>
</tr>
<tr>
<td>CC3</td>
<td>1.923e-8</td>
<td>1.223e-7</td>
<td>1.438e-5</td>
</tr>
</tbody>
</table>

### Table II. POLYNOMIAL COEFFICIENTS IN RELATION (14)

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(b_3)</th>
<th>(b_2)</th>
<th>(b_1)</th>
<th>(b_0)</th>
</tr>
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<tbody>
<tr>
<td>0.20</td>
<td>6.733</td>
<td>-5.890</td>
<td>1.4257</td>
<td>0.5536</td>
</tr>
<tr>
<td>0.25</td>
<td>6.9667</td>
<td>-5.945</td>
<td>1.4838</td>
<td>0.5716</td>
</tr>
<tr>
<td>0.30</td>
<td>7.4907</td>
<td>-6.1644</td>
<td>1.5560</td>
<td>0.5856</td>
</tr>
<tr>
<td>0.35</td>
<td>7.0533</td>
<td>-5.862</td>
<td>1.5379</td>
<td>0.6006</td>
</tr>
<tr>
<td>0.40</td>
<td>4.5266</td>
<td>-4.331</td>
<td>1.3084</td>
<td>0.6214</td>
</tr>
<tr>
<td>0.50</td>
<td>-5.040</td>
<td>1.704</td>
<td>0.2686</td>
<td>0.6866</td>
</tr>
</tbody>
</table>

### REFERENCES


