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Additive Noise Removal using a Nonlinear Hyperbolic PDE-based Model

Tudor Barbu
Institute of Computer Science of the Romanian Academy
Iasi, Romania
tudor.barbu@iit.academiaromana-is.ro

***Abstract*—A novel second-order partial differential equation (PDE) - based image restoration technique is proposed here. The considered smoothing method is based on a nonlinear hyperbolic differential model combined to a two-dimension filter kernel. The considered PDE model is well-posed and it is solved numerically by developing an explicit iterative finite difference method-based numerical approximation algorithm that is consistent to the combined PDE-based model and is converging fast to its weak solution. Our successful restoration tests and method comparison are also discussed.**

Keywords— image restroration; additive Gaussian noise; hyperbolic PDE scheme; filter kernel; finite differences; numerical approximation algorithm.

I. INTRODUCTION

The feature-preserving denoising and restoration still represents a very challenging image processing problem. Since the conventional filtering approaches and the linear partial differential equation (PDE) - based schemes may produce the unintended image blurring effect which affects the edges and other features [1], the nonlinear PDE models represent a considerably better solution for detail-preserving smoothing.

Second-order nonlinear PDE-based restoration models, which are inspired by the well-known anisotropic diffusion - based Perona-Malik scheme [2,3], and the PDE variational methods influenced by the TV-ROF Denoising model [4-7], provide a satisfactory noise removal, reduce considerably the blurring and preserve the essential details. Unfortunately, they also generate the undesirable staircasing effect [8].

This unintended effect can be avoided or alleviated by improving the second-order PDE denoising schemes or using nonlinear fourth order PDE restoration models, such as the influential You-Kaveh scheme [9]. The fourth-order diffusion - based algorithms overcome successfully the staircasing, but may produce over-filtering and some multiplicative (speckle) noise [10].

We have elaborated numerous nonlinear second and fourth order diffusion-based denoising techniques in our past papers [11-14]. In this work we propose an improved second-order PDE-based denoising approach that removes sucessfully the additive Gaussian noise generated by the image collection and transmission processes and overcomes or alleviates all the undesired effects.

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While the second order nonlinear PDE-based filtering methods obtained from the Perona-Malik model have a parabolic character, our restoration technique is based on a hyperbolic PDE model that provides much sharper edges and image details. It also has a compound character, incorporating a component that combines by convolution the evolving image to a 2D filter kernel. The nonlinear combined hyperbolic differential model is described in the next section.

Next, the considered PDE model is solved numerically using an iterative finite difference-based algorithm that is described in the third section. The restoration simulations performed by applying that algorithm and some comparisons with other approaches are described in the fourth section. The work ends with conclusion and a list of references.

II. NONLINEAR SECOND-ORDER HYPERBOLIC PDE MODEL

We consider here a combined nonlinear second-order diffusion-based model for image denoising. It contains a hyperbolic PDE with several boundary conditions, and also involves a conventional two-dimension filter. It is expressed as follows:

$$\begin{cases} \alpha \dfrac{\partial^2 u}{\partial t^2} + \beta \dfrac{\partial u}{\partial t} - \varphi(\|\nabla u * K(x,y)\|) \nabla \cdot \left(\delta_u(\|\nabla u\|) \nabla u\right) + \lambda (u - u_0) = 0 \\ u(x,y,0) = u_0(x,y), \quad \forall (x,y) \in \Omega \\ \dfrac{\partial u}{\partial t}(0,x,y) = u_1(x,y) \\ u(t,x,y) = 0, \quad \forall (x,y) \in \partial\Omega \end{cases} \tag{1}$$

where the parameters α, β, $\lambda \in (0,1]$, the image domain $\Omega \subseteq R^2$, u_0 represents the observed image and $K(x, y)$ is a 2D filter kernel. The first function used in this model has the following form:

$$\begin{cases} \varphi : (0,\infty) \to (0,\infty) \\ \varphi(s) = \eta^{k+1}\sqrt{\xi s^k + \zeta} \end{cases} \tag{2}$$

where $\eta, \xi, \zeta \in (0,4]$ and $k \in [0,2]$. The term that is based on it, $\varphi(\|\nabla u * K(x,y)\|)$, is introduced to control the speed of the diffusion process and enhance the image boundaries. If one considers a 2D Gaussian smoothing kernel, then we have:

$$K(x,y) = G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{3}$$

where the standard deviation $\sigma > 0$.

The edge-stopping, or diffusivity, function of this PDE-based denoising model is based on the evolving image and has the following form:

$$\delta_u : [0,\infty) \to [0,\infty), \delta_u(s) = \nu\left(\frac{\gamma(u)}{\varepsilon \ln(\gamma(u)) + \varsigma s^r}\right)^{\frac{1}{r+1}} \tag{4}$$

where $\nu \in (0,2)$, $\varsigma \in (1,4]$, $\varepsilon \in (0,1)$ and the conductance parameter is constructed as:

$$\gamma(u) = \|v\mu(\|\nabla u\|) + tw\| \tag{5}$$

where $v \in (0,3)$, $w \in (0,1)$ and $\mu(\cdot)$ returns the mean value.

The diffusivity function provided in (4) is properly modeled, satisfying the main conditions that are required by a successful diffusion [2,3]. Thus, it is positive, monotonic decreasing and convergent to zero ($\lim_{s\to\infty} \delta_u(s) = 0$).

The nonlinear second-order hyperbolic diffusion-based model (1) produces an effective additive noise removal and deblurring. Since it is also based on a second time derivative, it removes successfully the diffusion effect in the vicinity of the image boundaries, thus providing much sharper edges and better image details.

Also, the proposed PDE-based restoration model is well-posed. The existence of a weak and unique solution of (1) can be demonstrated under some certain assumptions. That solution, which corresponds to the recovered image, is identified by solving numerically the nonlinear PDE scheme. So, a finite difference-based numerical discretization scheme is proposed for the model, in the next section.

III. Numerical Approximation Algorithm

We consider a numerical approximation scheme using the finite-difference method [15], to solve the proposed differential model. Thus, we use a grid with space size h and time step size Δt and quantize the time and space coordinates as:

$$x = ih, y = jh, t = n\Delta t, i \in \{1,...,I\}, j \in \{1,...,J\}, n \in \{1,...,N\} \tag{6}$$

with $[Ih \times Jh]$ representing the dimension of the support image.

We can re-write the partial differential equation in (1) as following:

$$\alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \lambda(u - u_0) = \varphi(\|\nabla K * u\|)\left(\frac{\partial}{\partial x}(\delta_u(\|\nabla u\|)u_x) + \frac{\partial}{\partial y}(\delta_u(\|\nabla u\|)u_y)\right) \tag{7}$$

The left term in (7) is then discretized, by using finite differences [15], as:

$$\begin{aligned} &\alpha\frac{u_{i,j}^{n+\Delta t} + 2u_{i,j}^n - u_{i,j}^{n-\Delta t}}{(\Delta t)^2} + \beta\frac{u_{i,j}^{n+\Delta t} - u_{i,j}^n}{\Delta t} + \lambda(u_{i,j}^n - u_{i,j}^0) = \\ &u_{i,j}^{n+\Delta t}\frac{(\alpha + \beta\Delta t)}{(\Delta t)^2} + u_{i,j}^n\frac{2\alpha - \beta\Delta t + \lambda(\Delta t)^2}{(\Delta t)^2} - u_{i,j}^{n-\Delta t}\frac{\alpha}{(\Delta t)^2} - u_{i,j}^0\lambda \end{aligned} \tag{8}$$

Then, the right term is approximated by using central differences [15]. First, we compute $\varphi_{i,j} = \varphi(\|(u * K)_{i,j}\|)$ and $\delta_{i,j} = \delta_u(\|u_{i,j}\|)$, where:

$$\|u_{i,j}\| \approx \sqrt{\left(\frac{u_{i+h,j} - u_{i-h,j}}{2h}\right)^2 + \left(\frac{u_{i,j+h} - u_{i,j-h}}{2h}\right)^2} \tag{9}$$

The components $\frac{\partial}{\partial x}(\delta_u(\|\nabla u\|)u_x)$ and $\frac{\partial}{\partial y}(\delta_u(\|\nabla u\|)u_y)$ are discretized spatially as $\delta_{i+\frac{h}{2},j}(u_{i+h,j} - u_{i,j}) - \delta_{i-\frac{h}{2},j}(u_{i,j} - u_{i-h,j})$ and $\delta_{i,j+\frac{h}{2}}(u_{i,j+h} - u_{i,j}) - \delta_{i,j-\frac{h}{2}}(u_{i,j} - u_{i,j-h})$ respectively, where we have:

$$\delta_{i\pm\frac{h}{2},j} = \frac{\delta_{i\pm h,j} + \delta_{i,j}}{2}, \delta_{i,j\pm\frac{h}{2}} = \frac{\delta_{i,j\pm h} + \delta_{i,j}}{2} \tag{10}$$

We may use $h = \Delta t = 1$ and get the following implicit approximation algorithm:

$$\begin{aligned} &u_{i,j}^{n+1}(\alpha+\beta) + u_{i,j}^n(2\alpha - \beta + \lambda) - u_{i,j}^{n-1}\alpha - u_{i,j}^0\lambda = \varphi_{i,j} \\ &\left(\delta_{i+\frac{1}{2},j}(u_{i+1,j}^n - u_{i,j}^n) - \delta_{i-\frac{h}{2},j}(u_{i,j}^n - u_{i-1,j}^n) + \delta_{i,j+\frac{1}{2}}(u_{i,j+1}^n - u_{i,j}^n) - \delta_{i,j-\frac{1}{2}}(u_{i,j}^n - u_{i,j-1}^n)\right) \end{aligned} \tag{11}$$

It leads to the next explicit iterative numerical approximation scheme:

$$u_{i,j}^{n+1} = \frac{1}{\alpha+\beta}\left(\begin{aligned} &u_{i,j}^n\left(\beta - 2\alpha - \lambda - \varphi_{i,j}\left(\delta_{i+\frac{1}{2},j} + \delta_{i-\frac{1}{2},j} + \delta_{i,j+\frac{1}{2}} + \delta_{i,j-\frac{1}{2}}\right)\right) + \\ &u_{i+1,j}^n\delta_{i+\frac{1}{2},j}\varphi_{i,j} + u_{i-1,j}^n\delta_{i-\frac{h}{2},j}\varphi_{i,j} + u_{i,j+1}^n\delta_{i,j+\frac{1}{2}}\varphi_{i,j} + u_{i,j-1}^n\delta_{i,j-\frac{1}{2}}\varphi_{i,j} + u_{i,j}^{n-1}\alpha + u_{i,j}^0\lambda \end{aligned}\right) \tag{12}$$

The explicit iterative numerical approximation scheme (11) is stable and consistent to the nonlinear hyperbolic PDE-based model (1). It is converging rapidly to its weak solution representing the filtered image and has been successfully applied in our numerical experiments that are described in the next section.

IV. Denoising Experiments and Method Comparison

The proposed second order PDE-based filtering technique has been tested on numerous images corrupted with various levels of additive Gaussian noise. We have used some well-known image databases, such as the 3 volumes of the USC - SIPI database, in our experiments.

The performance of the proposed approach has been assessed using the Peak Signal to Noise Ratio (PSNR) similarity metric [16]. We have identified empirically a sequence of parameter values providing an optimal denoising:

$$\begin{aligned} &\lambda = 0.3, \alpha = 1.2, \beta = 0.7, \eta = 0.7, \xi = 1.3, \zeta = 1.4, k = 0.4, \\ &\nu = 0.5, \varepsilon = 0.4, \varsigma = 3.5, r = 2, v = 1.6, w = 0.25, N = 10 \end{aligned} \quad (13)$$

Our hyperbolic diffusion technique removes successfully the additive noise, while avoiding the multiplicative noise. It overcomes the blurring and also alleviates the blocky effect. It preserves the image details very well and has a low running time, because of its fast-converging numerical approximation scheme.

The proposed restoration framework outperforms classic 2D filters, by providing a better filtering and overcoming the unintented blurring effect. It also performs better than nonlinear second-order PDE and total variation models derived from Perona-Malik and TV-ROF Denoising schemes, because it provides much sharper details (edges and corners), given its hyperbolic character, and reduces the staircasing. Our method may represent a better smoothing solution even than the You-Kaveh-inspired fourth-order diffusion-based schemes, since it avoids the speckle noise and defines much better the boundaries and other features.

See several method comparison results in the following tables. Thus, the averaging PSNR values achieved by various PDE-based and classic filters on images corrupted by relatively moderate amounts of additive noise are registered in Table I, while the PSNR values obtained by those smoothing techniques on images corrupted by higher amounts of 2D Gaussian noise are displayed in Table II. Our second-order PDE-based denoising framework achieves better values (higher PSNRs) than other methods.

Some method comparison examples are described in the next figures. The original [256 x 256] *Tree* image from USC - SIPI database (Vol. 3) is displayed in Fig. 1 (a). It has been corrupted in (b) by a Gaussian noise with μ = 0.05 and *variance* = 0.02. The filtering result obtained by the proposed method is displayed in (c). The smoothing results achieved by various conventional filters are depicted on (d) – (f) and the output achieved by other PDE-based filtering models are displayed in (g) and (h). See the filtering results produced by these filters on a *Baboon* image affected more severely by additive noise (parameters μ = 0.10 and *var* = 0.08) in Fig. 2. Our model in (c) provides the best results here, too.

TABLE I. Average PSNR values (Moderate Noise)

Restoration Technique	Average PSNR value
This PDE-based model	28.3542 (dB)
2D Gaussian filter	23.3271 (dB)
Average filter	24.9826 (dB)
Wiener filter	25.4983 (dB)
Perona-Malik	26.1244 (dB)
TV-ROF model	27.8742 (dB)

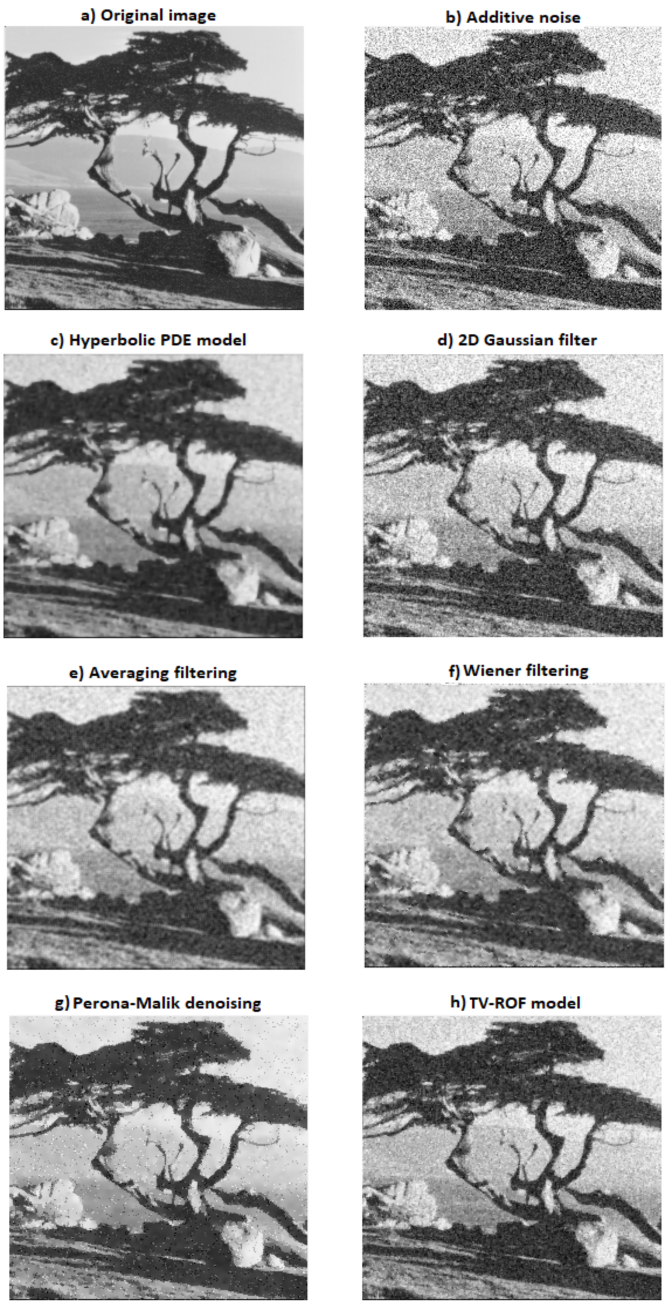

Fig. 1. Filtering results of PDE and non-PDE models for moderate noise

TABLE II. AVERAGE PSNR VALUES (HIGH AMOUNT OF NOISE)

Restoration Model	Average PSNR value
This hyperbolic PDE model	27.1512 (dB)
Gaussian filter	22.9710 (dB)
Average filter	24.1260 (dB)
Wiener 2D	25.2030 (dB)
Perona & Malik	25.5124 (dB)
TV Denoising	26.4133 (dB)

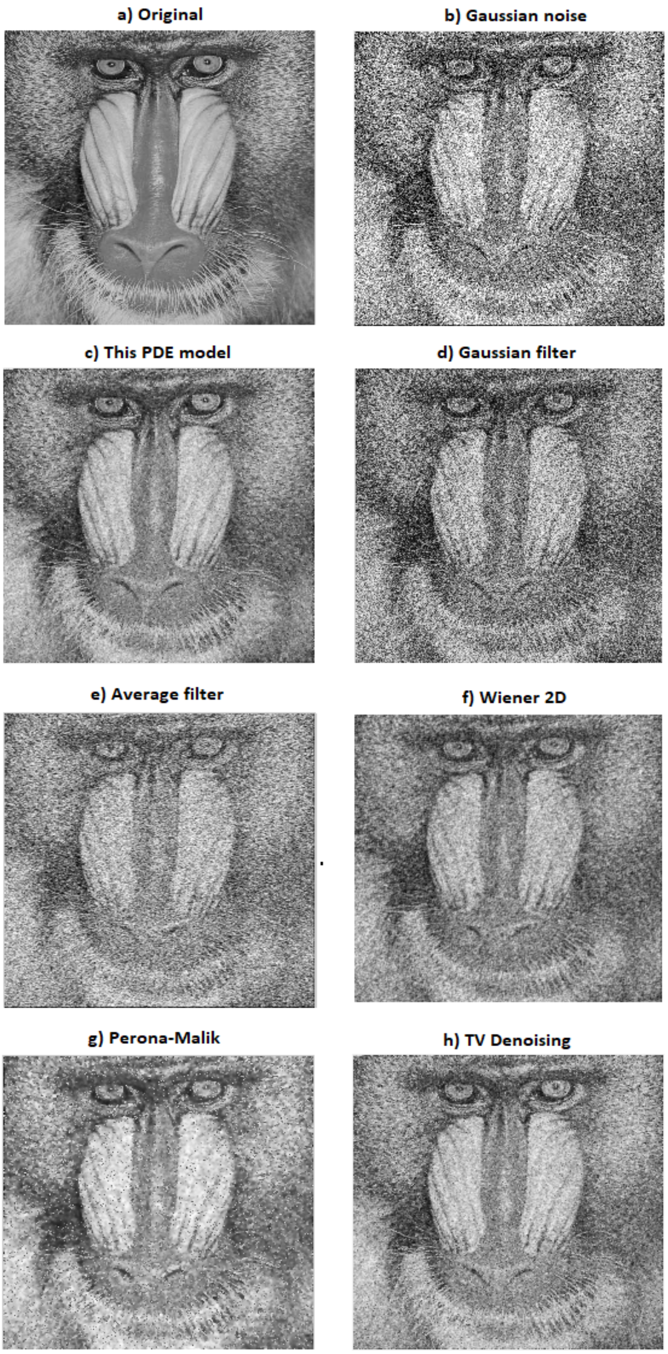

Fig. 2. Smoothing results of PDE and non-PDE filters (high level of noise)

V. CONCLUSIONS

We have proposed a nonlinear second order hyperbolic PDE-based framework for additive noise removal in this article. It represents a compound well-posed PDE model based on a nonlinear diffusion component that assures a detail-preserving image restoration and a component combining the evolving image to a filtering kernel, which has the role of controlling the speed of the diffusion process and defining the image edges.

The unique and weak solution of this hyperbolic model is computed by constructing a finite-difference method based iterative explicit numerical approximation scheme that is stable, consistent to the diffusion model and converges very fast to it.

Our image restoration technique provides an effective smoothing while avoiding the undesired effects. It has a strong detail-preserving character, since it produces much sharper edges because of the hyperbolic character of its equation. Thus, it outperforms not only the classic 2D filtering schemes, but also a lot of popular second-order PDE-based approaches, such as Perona-Malik model and the total variation based schemes.

The image filtering experiments and method comparison presented in this work illustrate the effectiveness of the developed diffusion-based restoration method. Given its strong edge-preserving character, the proposed smoothing approach could be successfully applied in the edge detection and image object detection areas. Effective image completion solutions may be also derived from this PDE restoration model.

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