THE THEOREMS ABOUT THE SYSTEM'S PRACTICAL STABILITY WITH THE MEASURABILITY OF THE PHASE SPACE

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Abstract. *The attributes and the proved theorems of the practical stability were formulated for the systems with the variable measurability. The determined criteria of the practical stability, which are suitable for the direct use as numerical algorithms of definition of the domain stability. On the base of there theorems the constructive problems' solutions of practical stability for the non-stationary linear systems of differential equations were cited.*
Keywords: *practical stability, phase space, variable measurability, criteria of the practical stability*

Introduction

The aim of the given scientific work is the investigation of the practical stability [1,2] with the variable measurability of the phase space [3,4].
For this system we consider limited time intervals and limited perturbation space of initial conditions. The stability task with the same features is called the chasks of practical stability.

The mathematical model of practical stability systems with the variable measurability

Let $\tau_1, \tau_2, ..., \tau_N$ is some segment's division $[T_0, T_1]$, where $\tau_j = \{t : t \in [t_{j-1}, t_j)\}$, $j = 1,2,...,N-1$, $\tau_N = \{t : t \in [t_{N-1}, t_N]\}$, $t_0 = T_0 < t_1 < t_2 < ... < t_{N-1} < t_N = T_1$.
Let us suppose, that on this segment the system dynamics is give as

$$\frac{dx^{(j)}(t)}{dt} = f^{(j)}\left(x^{(j)}(t), t\right),\ t \in \tau_j,\ j = \overline{1,N}, \quad (1)$$

$$x^{(j)}\left(t_{j-1}\right) = g^{(j)}\left(x^{(j-1)}\left(t_{j-1}-\right)\right),\ j = \overline{1,N}, \quad (2)$$

where $x^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, ..., x_{n_j}^{(j)}\right)^*$ – is the n_j-measurable vector of the phase coordinates, $f^{(j)}\left(x^{(j)}(t), t\right)$ – is the n_j-dimensions functions that satisfy the conditions of existence and unity of solution of the system (1) theorem when $t \in \tau_j$, $j = \overline{1,N}$, $g^{(j)}\left(x^{(j-1)}\left(t_{j-1}-\right)\right)$ – is the dimensions functions of n_j that give the change of system (1) of the phase condition measurability in the moments when t_{j-1}, $j = \overline{1,N}$, $g^{(1)}\left(x^{(0)}\left(t_0-\right)\right) = x^{(1)}\left(t_0\right)$, the symbol * – marks the transposition operation later on.
Let us suppose, that $f^{(j)}(0,t) = 0$ when $t \in \tau_j$, $g^{(j)}(0) = 0$, $j = \overline{1,N}$, it mean that the vectors of suitable dimensions with zero components.
Attribute 1. Unperturbed movement $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, of systems (1), (2) let us call $\left(G_0, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$-stable, if $x^{(j)}(t) \in \Phi_t^{(j)}$, when $t \in \tau_j$, $j = \overline{1,N}$, and $x^{(1)}(t_0) = x_0^{(1)} \in G_0$.
Attribute 2. Solution of system (1) and (2), $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, we will call $\left(G_0, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, \infty\right)$-asymptotically stable, if:

a) The conditions of attribute 1 come to be for some $T_1 < \infty$;
b) $\lim_{t \to \infty} \left\| x^{(N)}\left(t, x_0^{(1)}\right) \right\| = 0$ for some $x^{(1)}(t_0) = x_0^{(1)} \in G_0$.

In this paper under $\Phi_t^{(j)}$ we will understand the admissible multitudes of the conditions of vectors $x^{(j)}(t)$ if $t \in \tau_j$,

$$\left\| x^{(j)}(t) \right\| = (x^{(j)*} x^{(j)})^{\frac{1}{2}} = \left(\sum_{i=1}^{n_j} \left(x_i^{(j)}(t) \right)^2 \right)^{\frac{1}{2}}, \ j = \overline{1,N},$$

G_0 – the multitude of admissible initial conditions of system (1) and (2) in case $t = t_0$.

For the construction of the effective methods of the checking the quality of practical stability of dynamic system let us consider the multitude of the initial conditions of type

$$G_0 = \{x^{(1)} : x^{(1)T} B x^{(1)} < c^2\},$$

where B is additionally denoted matrix with dimension $n_1 \times n_1$, $c > 0$ – is some stable value.

Attribute 3. Unperturbed movement of the system (1), (2) $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, we will call $\left(c, B, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$-stable, if $x^{(j)}(t) \in \Phi_t^{(j)}$ when $t \in \tau_j$, $j = \overline{1,N}$, for all the initial conditions

$$x^{(1)}(t_0) = x_0^{(1)} \in \{x_0^{(1)} : x_0^{(1)*} B x_0^{(1)} < c^2\}.$$

If $G_0 = \{x_0^{(1)} : x_0^{(1)*} x_0^{(1)} < \lambda^2\}$, than unperturbed movement of the system (1), (2), $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, that satisfy the attribute 3, we will call stable $\left(\lambda, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$.

The criteria of practical stability we will formulate for two type of multitudes:

$$\Phi_t^{(j)} = \{x^{(j)} : \psi_j(x^{(j)}, t) \le 1\}, \ t \in \tau_j, \ j = \overline{1,N},$$
$$\Gamma_t^{(j)} = \{x^{(j)} : \left| l_s^{(j)*}(t) x^{(j)} \right| \le 1, s = 1,2,...,M_j\},$$
$$t \in \tau_j, \ j = \overline{1,N},$$

where $l_s^{(j)}(t)$ – is nonseparable vector-functions with size n_j; $\psi_j(x^{(j)}, t)$ – is scalar functions that are nonseparable in the totality of arguments when $t \in \tau_j$ together with its partial derivatives conforming to the component of vector $x^{(j)}$; $\Phi_t^{(j)}$ – is the closed rounded multitudes for some $t \in \tau_j$, that include the interior point $x^{(j)}(t) = 0$, $j = \overline{1,N}$.

Let us denote $a^{(j)*} x^{(j)} = \alpha_j$ – are the hyperplanes in $n^{(j)}$-measurable Euclidean spaces, $j = \overline{1,N}$.

We will use the Liapunov's functions denotation, supposing its continuity together with partial похідними on the multitudes of designation, relations

$$\left| a^{(j)*} x^{(j)} \right| \le \sqrt{c^2 a^* \left(B_j \right)^{-1} a},$$
$$x^{(j)} \in \{x^{(j)} : x^{(j)*} B_j x^{(j)} \le c_j{}^2\},$$

or

$$\left| a^{(j)*} x^{(j)} \right| \le \sqrt{x^{(j)*} B_j x^{(j)} a^{(j)*} B_j^{-1} a^{(j)}},$$

also the Reley's irregularity

$$\lambda_{j\min} \le \frac{x^{(j)*} B_j x^{(j)}}{x^{(j)*} x^{(j)}} \le \lambda_{j\max},$$

where $\lambda_{j\min}$, $\lambda_{j\max}$ – is respectively minimum and maximum own meaning of additional denoted matrix B_j, $j = \overline{1,N}$.

Theorem 1. If for the system (1) and (2) will be find the Liapunov's additional specify functions $V_j(x^{(j)}, t)$, that satisfy the conditions:

$$\{x^{(j)} : V_j(x^{(j)}, t) < 1\} \subset \Phi_t^{(j)}, \ t \in \tau_j, \ j = \overline{1,N}; \quad (3)$$

$$\left(\frac{dV_j(x^{(j)}, t)}{dt} \right)_{(1),(2)} \le 0$$

when

$$x^{(j)} \in \{x^{(j)} : V_j(x^{(j)}, t) < 1\}, \ t \in \tau_j, \ j = \overline{1,N}; \quad (4)$$

for some $s \in \{1,2,...,N\}$

$$\sum_{j=1}^{s-1} (V_j(x^{(j)}(t_j-), t_j) - V_{j+1}(x^{(j+1)}(t_j), t_j)) \ge 0; \quad (5)$$

$$G_0 \subset \{x^{(1)} : V_1(x^{(1)}, t_0) < 1\}, \qquad (6)$$

than unperturbed movement $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, of the systems (1) and (2) is $\left(G_0, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$-stable.

Proof (by contradiction). Let us suppose that the conditions (3)–(6) are realized but will be find $1 \le j_0 \le N$ and this meaning $\bar{t} \in \tau_j$, when $x^{(j_0)}(\bar{t}) \notin \Phi_t^{(j_0)}$. According to the condition (3) and to the continuity of Liapunov's function $V_{j_0}(x^{(j_0)}, t)$ in the point $\bar{t}$ will be realized the non-equation $V_{j_0}(x^{(j_0)}(\bar{t}), \bar{t}) \ge 1$. Besides, according to the (4) condition on the trajectories of system (1) and (2), we have

$$\sum_{j=1}^{j_0-1} \int_{t_{j-1}}^{t_j} \left(\frac{dV_j(x^{(j)}, t)}{dt} \right)_{(1),(2)} dt + V_{j_0}(x^{(j_0)}(\bar{t}), \bar{t}) - -V_{j_0}(x^{(j_0)}(t_{j_0-1}), t_{j_0-1}) \le 0 .$$

.

Therefore

$$V_1\left(x^{(1)}(t_1 -), t_1\right) - V_1\left(x^{(1)}(t_0), t_0\right) + +V_2\left(x^{(2)}(t_2 -), t_2\right) - V_2\left(x^{(2)}(t_1), t_1\right) + ... + +V_{j_0-1}\left(x^{(j_0-1)}(t_{j_0-1} -), t_{j_0-1}\right) - V_{j_0-1}\left(x^{(j_0-1)}(t_{j_0-2}), t_{j_0-2}\right) + + ... + V_{j_0}\left(x^{(j_0)}(\bar{t}), \bar{t}\right) - V_{j_0-1}\left(x^{(j_0)}(t_{j_0-1}), t_{j_0-1}\right) \le 0 .$$

Therefore

$$\sum_{j=1}^{j_0-1}\left(V_j\left(x^{(j)}(t_j -), t_j\right) - V_{j+1}\left(x^{(j+1)}(t_j), t_j\right)\right) + +V_{j_0}\left(x^{(j_0)}(\bar{t}), \bar{t}\right) - V_1\left(x^{(1)}(t_0), t_0\right) \le 0 .$$

Hence, using the (5) condition we have $V_1(x^{(1)}(t_0), t_0) \ge 1$, that contradict to the (6) condition. Therefore our supposition is incorrect. The theorem 1 is proved.

Theorem 2. If for the system (1) and (2) will be find the Liapunov's additional specify functions $V_j(x^{(j)}, t)$, $j = \overline{1,N}$, that satisfy the conditions (3)–(5) and

$$\{x^{(1)} : x^{(1)*} B x^{(1)} < c^2\} \subset \{x^{(1)} : V_1(x^{(1)}(t_0), t_0) < 1\}, \qquad (7)$$

than unperturbed movement $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, of the systems (1) and (2) is $\left(c, B, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$-stable.

The proof of the theorem can be made analogously the scheme of the theorem 1.

Theorem 3. Let us suppose that system (1), (2) is $\left(c, B, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, T_1\right)$ stable, the functions $g^{(j)}\left(x^{(j-1)}\left(t_{j-1} -\right)\right)$ have the same dimensions $n_j = n$, there are the reciprocal functions $\psi^{(j)} = \left(g^{(j)}\right)^{-1}$, e.g. $x^{(j-1)}\left(t_{j-1} -\right) = \psi^{(j)}\left(g^{(j)}\left(x^{(j-1)}\left(t_{j-1} -\right)\right)\right)$, $j = \overline{1,N}$. Than will be find Liapunov's functions $V_j(x^{(j)}, t)$, $t \in \tau_j$, $j = \overline{1,N}$, that satisfy the conditions of the theorem 2.

Proof. Let us consider the following functions:

$$V_j(x^{(j)}, t) = \frac{1}{c^2} F^{(j)*}(x^{(j)}, t) B F^{(j)}(x^{(j)}, t), \qquad (8)$$

when

$$F^{(j)}(x^{(j)}, t) = \varphi^{(1)}\left(\psi^{(2)}\left(\varphi^{(2)}\left(...\left(\psi^{(j)}\left(\varphi^{(j)}\left(x^{(j)}, t, t_{j-1}\right), t_{j-1}, t_{j-2}\right)..., t_2, t_1\right), t_1, t_0\right)\right)\right),$$

$$x^{(j)}(t_{j-1}) = \varphi^{(j)}(x^{(j)}, t, t_{j-1}),\ t \in \tau_j,\ j = \overline{1,N} .$$

These functions are additional specified, because $V_j(0, t) = 0$ and $V_j(x^{(j)}, t) > 0$ when $\left\|x^{(j)}\right\| \ne 0$, $t \in \tau_j$, (decision (1) and (2) exists and is unique when $t \in \tau_j$), $j = \overline{1,N}$. If consider some trajectory of the system (1), (2), built conforming to the initial condition $x^{(1)}(t_0) = x_0^{(1)}$, choose some $t \in \tau_j$ point $x^{(j)}(t)$ on this trajectory when, than

$$F^{(j)}(x^{(j)}, t) = x_0^{(1)},\ j = \overline{2,N} \text{ and } \varphi^{(1)}\left(x^{(1)}(t), t, t_0\right) = x_0^{(1)} .$$

Therefore $\left(\dfrac{dV_j(x^{(j)},t)}{dt}\right)_{(1),(2)} = 0$ when $t \in \tau_j$, because $V_j(x^{(j)},t)$ accumulates the stable meanings on the trajectories $x^{(j)}(t)$, $t \in \tau_j$, of the system (1), (2):

$$V_1(x^{(1)}(t),t) = \frac{1}{c^2}\left(x_0^{(1)}\right)^* Bx_0^{(1)},\ t \in \tau_j,\ j = \overline{1,N}.$$

So $V_j(x^{(j)},t)$, $j = \overline{1,N}$, satisfy conditions (5).

Let us prove that the conditions (3) realize. Suppose that system (1), (2) is $\{c, B, \Phi_t^{(1)}, \Phi_t^{(2)},..., \Phi_t^{(N)}, T_0, T_1\}$-stable, but there is such interval τ_{j_0}, $(1 \le j_0 \le N)$ and such meaning $\overline{x}^{(j_0)} \notin \Phi_t^{(j_0)}$ for which the non-equation is correct

$$V_{j_0}\left(\overline{x}^{(j_0)},t\right) = \frac{1}{c^2} F^{(j_0)*}\left(\overline{x}^{(j_0)},t\right) B F^{(j_0)}\left(\overline{x}^{(j_0)},t\right) < 1.$$

From the last non-equation imply that for the initial point $x_0^{(1)} = F^{(j_0)}\left(x^{(j_0)}(t),t\right)$ of the trajectory that passes through a point $\overline{x}^{(j_0)}$ a non-equation is correct

$$\frac{1}{c^2} x_0^{(1)*} B x_0^{(1)} < 1.$$

From here $x_0^{(1)} \in \left\{x^{(1)} : x^{(1)*} B x^{(1)} < c^2\right\}$, therefore from the system (1), (2) stability $\left(c, B, \Phi_t^{(1)}, \Phi_t^{(2)},..., \Phi_t^{(N)}, T_0, T_1\right)$ imply $\overline{x}^{(j_0)} \in \Phi_t^{(j_0)}$, that contradict to our supposition.

Also the condition (5) realizes because

$$\left\{x^{(1)} : x^{(1)*} B x^{(1)} < c^2\right\} = \left\{x^{(j)} : V\left(x^{(j)},t\right) < 1,\ t \in \tau_j\right\} = \\ = ... = \left\{x^{(N)} : V\left(x^{(N)},t\right) < 1,\ t \in \tau_N\right\}.$$

The theorem 3 is proved.

Consequence 1. If for system (1), (2) it is possible to find additionally denoted Liapunov's functions $V_j(x^{(j)},t)$, $t \in \tau_j$, $j = \overline{1,N}$, that satisfy the conditions (3)–(5) and

$$\left\{x^{(1)} : \left\|x^{(1)}\right\| < \lambda\right\} \subset \left\{x^{(1)} : V_1\left(x^{(1)},t_0\right) < 1\right\}, \tag{9}$$

than unperturbed movement $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1,N}$, of the system (1), (2) is $\{\lambda, \Phi_t^{(1)}, \Phi_t^{(2)},..., \Phi_t^{(N)}, T_0, T_1\}$ – stable.

Consequence 2. Let the system (1), (2) $\{\lambda, \Phi_t^{(1)}, \Phi_t^{(2)},..., \Phi_t^{(N)}, T_0, T_1\}$ - is a stable, function $g^{(j)}\left(x^{(j-1)}\left(t^{(j-1)}-\right)\right)$ of the same size $n_j = n$, $j = \overline{1,N}$, there are the inverse functions $\psi^{(j)} = \left(g^{(j)}\right)^{-1}$, e.g.

$$x^{(j-1)}\left(t_{j-1}\right) = \psi^{(j)}\left(g^{(j)}\left(x^{(j-1)}\left(t_{j-1}\right)\right)\right).$$

Then will be find Liapunov's functions $V_j(x^{(j)},t)$, $t \in \tau_j$, $j = \overline{1,N}$, that satisfy the conditions (3)–(5), (9).

Proofs of the consequences 1 and 2 may be made according to the proving schemes of theorems 2 and 3. For the consequence 2 it is useful to choose the Liapunov's functions in the form of:

$$V_j(x^{(j)},t) = \frac{1}{c^2} F^{(j)*}(x^{(j)},t) F^{(j)}(x^{(j)},t),\ t \in \tau_j,\\ j = \overline{1,N}, \tag{10}$$

where $F^{(j)}(x^{(j)},t)$ – are the function denoted above.

Theorem 4. If it is possible to find for the system (1), (2) additionally denoted Liapunov's functions $V_j(x^{(j)},t)$, $t \in \tau_j$, $j = 1,2,...,N-1$, $\tau_N = \{[t_{N-1},\infty)\}$, that satisfy the conditions:

$$\left\{x^{(j)} : V_j\left(x^{(j)},t\right) < 1\right\} \subset \Phi_t^{(j)},\ t \in \tau_j,\ j = \overline{1,N}; \tag{11}$$

function $\left(\dfrac{dV_j(x^{(j)},t)}{dt}\right)_{(1),(2)}$ – denoted negative on the multitudes $\left\{x^{(j)} : V_j\left(x^{(j)},t\right) < 1\right\}$;

$$G_0 \subset \{x^{(1)} : V_1(x^{(1)}, t) < 1\} \quad (12)$$

There are the nonseparable functions $V_j^1(x^{(j)})$, $j = \overline{1, N}$, for which the relations realise

$$V_j(x^{(j)}, t) \le V_j^1(x^{(j)}), \quad (13)$$

when $x^{(j)} \in \{x^{(j)} : V_j(x^{(j)}, t) < 1\}$, $t \in \tau_j$, $V_j^1(0) = 0$, $j = \overline{1, N}$, then unperturbed decision $x^{(j)}(t) = 0$, $t \in \tau_j$, $j = \overline{1, N}$, of the system (1), (2) is asymptomatic $(G_0, \Phi_t^{(1)}, \Phi_t^{(2)}, ..., \Phi_t^{(N)}, T_0, \infty)$-stable.

Cite the constructive decisions of tasks of the practical stability for the linear nonstationary systems of the differential equations

$$\frac{dx^{(j)}(t)}{dt} = A_j(t) x^{(j)}(t), \; t \in \tau_j, \quad (14)$$

with the conditions of the phase space measurability change

$$x^{(j)}(t_{j-1}) = C_j x^{(j-1)}(t_{j-1} -), \quad (15)$$

where $A_j(t)$- is a quadratic matrix of the order n_j, $x^{(j)}(t)$ – n-measurable vector of the phase condition; C_j – is rectangular stable matrix of the size $n_j \times n_{j-1}$, $j = \overline{1, N}$. Where we will assume that $C_1 = E_1$ – is the identity matrix of the order n_1; $x^{(0)}(t_0 -) = x_0^{(1)}$ – is the initial condition of the system (14).

Theorem 5. For $\{\lambda, \Gamma_t^{(1)}, \Gamma_t^{(2)}, ..., \Gamma_t^{(N)}, T_0, T_1\}$-stability of system (14) with conditions of (15) it is enough that the following conditions will be realize

$$\lambda^2 \le \inf_{t \in \sigma_1} \min_{s = 1, 2, ..., M_1} \left(l_s^{(1)^*}(t) Q_1^{-1}(t) l_s^{(1)} \right)^{-1}, \quad (16)$$

$$\mu_j^2 \le \inf_{t \in \sigma_j} \min_{s = 1, 2, ..., M_j} \left(l_s^{(j)^*}(t) Q_j^{-1}(t) l_s^{(j)} \right)^{-1}, \; j = 2, ..., N$$

$$G_0 = \{x^{(1)*} x^{(1)} < \lambda^2\} \subset \Gamma_{t_0}^{(1)},$$

$$G_j = \{(C_j x^{(j)}(t_j -))^* C_j x^{(j)}(t_j -) < \mu_{j+1}^2\} \subset \Gamma_{t_j}^{(j+1)},$$

$$j = \overline{2, N-1},$$

where $x^{(j)}(t)$ – is decision (14) when $t \in \sigma_j$, that satisfies the initial condition (15), $j = \overline{1, N}$, and $x^{(1)}(t_0) = x_0^{(1)} \in G_0$.

Proof. Let us choose Liapunov's functions in form

$$V_j(x^{(j)}, t) = \frac{1}{\mu_j^2} x^{(j)*} Q_j(t) x^{(j)}, \; t \in \tau_j, \; j = \overline{1, N},$$

where $Q_j(t) = X_j^*(t_{j-1}, t) X(t_{j-1}, t)$ when $t \in \tau_j$, $\mu_1 = \lambda$, $\mu_2, ..., \mu_N$ – are such that the conditions (17) realize, $X(t, t_{j-1})$– are matrix decisions of Koshi's tasks

$$\frac{dX_j(t, t_{j-1})}{dt} = A_j(t) X_j(t, t_{j-1}),$$

$$X_j(t_{j-1}, t_{j-1}) = E_j, \; t \in \tau_j,$$

E_j – are the unit matrixes of the order n_j, $j = \overline{1, N}$.

Using the conditions (16) –(19) it is easy to convince that all the conditions of theorem1 realize. Theorem 5 is proved.

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