7th International Conference on DEVELOPMENT AND APPLICATION SYSTEMS
Suceava, Romania, May 27 – 29, 2004

OPTIMIZING OBSERVATION PARAMETERS IN THE ESTIMATING PROCESS

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Abstract. *In this paper we are continue to study the problems of minimax estimation linear functional under solutions systems of equations with partial derivations. New systems and new observe operators are considerate. New results are obtained in case unknown functions belong to Hilbert spaces with special metrics.*
Keywords: *minimax estimation, estimation error, functional minimization.*

Introduction

In this paper we consider the problem of minimizing a priori error of estimation of the state of systems described by linear parabolic equations. The solution of this problem we are using for the optimal constrain the parameters of the observer.

Mathematical model of the problem

Let $\Omega \subseteq R^n$ be an open bounded set (domain) with sufficiently regular boundary S. Consider the sets $Q = \Omega \times (t_0, t_1)$, $\Sigma = S \times (t_0, t_1)$, where (t_0, t_1) is an open interval in R^1. Let functions $a_{ij}(t,x),\ i = \overline{1,m}, j = \overline{1,m}$ be defined on Q such that $a_{ij}(t,x) \in L_\infty(Q),\ \sum_{i,j=1}^{m} a_{ij}(t,x)\xi_i\xi_j \ge \alpha \sum_{i=1}^{m} \xi_i^2$, where $\alpha > 0$ for all $\xi_i \in R^1$ almost everywhere in Q.

Consider the Sobolev space $W_2^1(\Omega)$ - of generalized functions. Denote by H_+ the closure of the space. Define a continuous bilinear form $a(t,\varphi,\phi) = \sum_{i,j=1}^{m} \int_\Omega a_{ij}(t,x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx$ on the elements $\varphi, \phi \in W_2^1(\Omega)$. The bilinear form is corresponding by the following continuous linear operator $A(t)\varphi = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial \varphi}{\partial x_j})$.

Under above assumptions, in the space $L_2(t_0, t_1, H_+)$ the equation

$$\frac{\partial \varphi}{\partial t} + A(t)\varphi = f_1,\ \varphi(t_0, x) = f_0, \qquad (1)$$

for arbitrary $f_1 \in L_2(t_0, t_1, W_2^{-1}(\Omega))$, $f_0 \in L_2(\Omega)$ has a unique solution φ. The equation (1) describes the behavior of some dynamical system.

We observe a vector $y = (y_1, y_2, \ldots, y_r)$ where

$$y_k = \int_\Omega l_k(t,\xi)\varphi(t,\xi)d\xi + f_k^{(2)}(t),\ k = \overline{1,r} \qquad (2)$$

The functions $l_k \in L_2(t_0, t_1, W_2^{-1}(\Omega))$ are controlled, they are supposed to be known for arbitrary $f_k^{(2)} \in L_2(t_0, t_1)$, and they describe the way we observe the dynamical system.

We consider a problem of how to estimate the state of the dynamical system (1) and then to minimize the estimation error by suitable choice of the control functions $l_k(t,x),\ k = \overline{1,r}$.

To solve the problem we need some additional (a priory) information about perturbations affecting the observer and the system under investigation.

Suppose, that the perturbations f_0, f_1, $f_k^{(2)}, k = \overline{1,r}$ belong to an ellipsoid in the space

$$L_2(t_0, t_1, W_2^{-1}(\Omega)) \times L_2(\Omega) \times \prod_{k=1}^{r} L_2(t_0, t_1),$$

that is

$$(Q_0 f_0, f_0) + \int_{t_0}^{t_1} (Q_1 f_1, f_1)_0 dt + \sum_{i=1}^{r} \int_{t_0}^{t_1} Q_i^{(2)}(t)(f_i^{(2)}(t))^2 dt \le 1 \quad (3)$$

where $Q_0(x)$ and $Q_i^{(2)}(t), i=\overline{1,r}$ are given additional functions on the set Ω and the interval (t_0, t_1), respectively, and Q_1 is a given symmetrical additional defined operator.

Then we can solve the problem of minimax estimation of a continuous linear functional defined on the states of dynamical system (1) with observations (2), and thereafter we can determine the estimation error (see [1]).

Consider solutions $z(t,x)$ and $p(t,x)$ of the following set of equations

$$\begin{cases} -\frac{\partial z}{\partial t} + A^*(t)z = l(t) - \sum_{k=1}^{r} Q_k^{(2)}(l_k(t), p)_0, \\ \qquad z(t_1, x) = 0, \\ \frac{\partial p}{\partial t} + A(t)p = Q_1^{-1} z, \\ \qquad p(t_0, x) = Q_0^{-1}(x) z(t_0, x). \end{cases} \quad (4)$$

Using the solutions (4), the minimax estimation error for a continuous linear functional

$$l(\varphi) = \int_{t_0}^{t_1} (l(t), \varphi) dt, \; l \in L_2(t_0, t_1, W_2^{-1}(\Omega)), \quad (5)$$

defined on the states of dynamical system, can be represent as a functional

$$\varepsilon^2 = \int_{t_0}^{t_1} (l(t), p) dt. \quad (6)$$

Consider a vector $g = (l_1, l_2, \ldots, l_r)$ from the set $V_g = \{\phi \in L_2^r(t_0, t_1, L_2(\Omega)) : \|\phi\| \le 1\}$. Introduce a functional

$$I(g) = \varepsilon^2(g) = \int_{t_0}^{t_1} (l(t), p(g)) dt \quad (7)$$

defined on the set V_g and consider the problem of minimizing the functional.

Main results

Now we are interested to know when the problem has solutions and what necessary conditions are satisfied by the solutions.

Theorem 1. The set $\arg\inf(I(g))$ is nonempty. If $\overline{g} \in \arg\inf(I(g))$, then for an arbitrary $g \in V_g$

$$\sum_{k=1}^{r} \int_{t_0}^{t_1} Q_k^{(2)}(\overline{l_k}, p)_0 ((l_k(t) - \overline{l_k}(t)), p) dt \le 0. \quad (8)$$

Proof. To prove the existence of solutions of the problem, we show that for the functional (7) some minimizing sequence converges. The embedding $H_+ \subset L_2$ is compact, and the observation operator is bounded in the metric of the corresponding space. Hence, the weak convergence of a minimizing sequence in V_g implies the strong convergence of corresponding solutions of the system (4) in the space H_+.

To obtain the necessary conditions (8), calculate in explicit form the Gateaux derivative of the functional (7). Now, transforming the expression for the $\frac{d}{d\varepsilon} I(\overline{g} + \varepsilon g)|(\varepsilon = 0) = 0$, we obtain the statement of the theorem.

Now consider the case when the state of the system can be observed only on the boundary of the domain Q. Suppose that we can observe an element $y = \int_{t_0}^{t_1} \frac{\partial \varphi}{\partial \eta_A} |\Sigma g(t) dt + f_2(x)$ of the Hilbert space $L_2(\Gamma)$, where $\varphi(t,x)$ is the solution of (1) when $f_0 = 0$ on Σ.

If the set G is given as follows

$$G = \{\overline{f} : \int_\Omega g_0^2(x) f_0^2(x) dx + \int_Q g_1^2(t,x) f_1^2(t,x) dx dt + \int_\Omega g_2^2(x) f_2^2(x) dx \le 1\}, \quad (9)$$

it can be show (see [1]), that the error of estimation of the linear functional

$$l(\varphi) = \int_Q l(t,x) \varphi(t,x) dx dt$$

can be represented as

$$\sigma^2 = \int_Q l(t,x) p(t,x) dx dt,$$

where the function $p(t,x)$ satisfied conditions

$$\begin{cases} -\frac{\partial z}{\partial t}+A^*(t)z=l, z(t_1,x)=0, \\ \qquad z|\Sigma=-C^*(g)\Lambda q_2^2 C(g)\frac{\partial p}{\partial \eta_A}|\Sigma, \\ \frac{\partial p}{\partial t}+A(t)p=q_1^{-2}z, \\ \qquad p(t_0,x)=q_0^{-2}(x)z(t_0,x), p|\Sigma=0. \end{cases}$$

Consider again minimizing the functional $I(g)=\sigma^2$ on the set $V(g)$.

Theorem 2. The set $\arg\inf(I(g))$ is nonempty. If $\bar{g}\in\arg\inf(I(g))$, then for an arbitrary $g\in V_g$,

$$\int_\Gamma (q_2^2(x)C(\bar{g})\frac{\partial p}{\partial \eta_A}|\Sigma)(C(g-\bar{g})\frac{\partial p}{\partial \eta_A}|\Sigma)dx\le 0$$

where $p=p(t,x)$ is determined from the solution of the system when $g=\bar{g}(t)$.

Proof. The most difficult point in proving the theorem is to show that the embedding of H_+ into $L_2(\Omega)$ if compact. In our case $p(*,x)\in H_+=H^{3/2}(\Omega)$ and by the trace theorem $\frac{\partial p}{\partial \eta_A}|\Sigma\in L_2(\Gamma)$, hence Sobolev`s embedding theorem is not application here. But, if $g_n\to\bar{g}$ weakly in $L_2(t_0,t_1)$, then the sequence $\{C(g_n)\}$ of operators, even of witch is a completely continuous operator, converges uniformly (in the norm of the space $\Im(L_2(\Sigma),L_2(\Gamma))$), to a completely continuous operator. Hence, we can perform passing to the limit in the corresponding set of equations. The theorem is proved.

Now consider the case of a single observation. Suppose that at time t_1 we observe the following element $y=C(m)\varphi(t_1)+f_2$,

of the space $L_2(\Omega)$, where the operator $C(m)$ is given by the formula

$$C(m)\psi=\int_\Omega m(x,\xi)\psi(\xi)d\xi .$$

If the domain G is given by (9), then the error of estimation of continuous linear functional

$$l(\varphi)=\int_Q g(t,x)\varphi(t,x)dxdt+\int_\Omega g_1(x)\varphi(t_1,x)dx$$

can be obtained in the form

$$\sigma^2=\int_Q g(t,x)p(t,x)dxdt+\int_\Omega g_1(x)p(t_1,x)dx ,$$

where the function $p(t,x)$ satisfied the following set of equations

$$\begin{cases} -\frac{\partial z}{\partial t}+A^*(t)z=g(t,x), \\ \qquad z(t_1,x)=q_1(x)+C^*(m)q_2^2C(m)p(t_1,x), \\ \qquad \frac{\partial z}{\partial \eta_{A^*}}|\Sigma=0, \\ \frac{\partial p}{\partial t}+A(t)p=q_1^{-2}z, \\ \qquad p(t_0,x)=q_0^{-2}(x)z(t_0,x), \\ \qquad \frac{\partial p}{\partial \eta_A}|\Sigma=q_0^{-2}z|\Sigma. \end{cases}$$

Analogously to the above considered cases, we can show that the set of solutions of the problem of minimizing a priory estimation error is nonempty, and we can obtain some necessary conditions which are satisfied by the solutions.

Examples

Using the Theorem 1, we can solve some practical problem. For example, suppose that the observation vector is $y=\int_{t_0}^{t_1} g(t)\varphi(t,x)dt$,

where $\varphi(t,x)$ is the solution of equation

$$\frac{\partial\varphi}{\partial t}+A(t)\varphi=f_1,\varphi(t_0,x)=f_0(x),\varphi|_\Sigma=\bar{f}_0(t,x).$$

Let the control function $g(t)$ belong to the set $V_g=\{\psi:\psi\in L_2(t_0,t_1),|\psi|\le 1\}$.

Then (8) can be rewritten as

$$\int_\Omega Q_2^{-1}(x)\int_{t_0}^{t_1}\bar{g}(t)p(t,x)dt\int_{t_0}^{t_1}[g(t)-\bar{g}(t)]p(t,x)dtdx\le 0$$

Assuming that $l(t,x)\ge 0$ on Q we obtain $\bar{g}(t)=\pm 1$.

References

[1] Наконечный, А.Г. Минимаксное оценивание функционалов от решений вариационных уравнений в гильбертовых пространствах. Киев:1985г.,-83с.