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SYSTEMS ANALYSIS UNDER FRACTAL NUMERICAL TECHNICS

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Abstract. *In this paper are revues the iterative techniques, which conduct to the fractal shapes and are involved into complexity theory treatment. Is performed a succinct presentation of some specific problems about the chaos hidden in simply relations and is shown few original ideas and result in this way. All items treat classic complexity problems that are concern into new perspective. The original results are obtained under own software applications, which are development in modelling and simulating lab.*
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Introduction

Many methods in the numeric calculus resolving nonlinear algebraic or differential equation but ignore some behaviour of these problems. Since fractal analysis developing were identified a lot of strange features in the solution of same equations.

A very interesting phenomenon occurs in the solution of the following set of nonlinear differential equations called the Loretz system [1]:

$$\frac{dx}{dt} = -10(x - y)$$
$$\frac{dy}{dt} = -xz + rx - y \qquad (1)$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

This system arises from problems related to fluid convection and to weather forecasting. When the r parameter lies in the $24.7<r<145$ interval, the solution does not converge to a fixed point in the $t\to\infty$ limit, nor is there a limit cycle, but the solution keeps moving around in a finite area. The limit set of the orbit at $t\to\infty$ is generally called the attractor. It has been confirmed numerically that the Lorentz attractor system has infinitely many foldings.

Another strange attractors have been found in many systems with few degrees of freedom. The following system, called the Rössler system [1], is famous for showing that chaos can be produces with only one nonlinear term:

$$\frac{dx}{dt} = -(y + z)$$
$$\frac{dy}{dt} = x + 0.2y \qquad (2)$$
$$\frac{dz}{dt} = 0.2 - 5.7z + xz$$

Attractors of ordinary differential equations with the degree of freedom less than 2 are limited to either a fixed point or a limit cycle, and have proved not to be strange. However, even in system with only two variables, chaos can be found if the system evolves discretely. A good example in this sense is the strange attractor of the Hennon map. The equations system in this case is:

$$x_{n+1} = 1 - ax_n^2 + by_n$$
$$y_{n+1} = x_n \qquad (3)$$

Strange attractors in systems of ordinary differential equations also usually have fractal properties. By imagining a plane in the phase space and observing only the points where the orbits pass through the plane, the dynamical systems can be reduced to a discrete map called the Poincaré map. The Poincaré map of the Rössler system, like the Henon map, is self-similar and the Rössler attractor is also fractal.

Strange attractors

Let us consider a simple nonlinear map called *logistic map* or *bifurcation map*:

$$p_{n+1} = rp_n(1-p_n), \quad 0 \le r \le 4 \tag{4}$$

This is an example of iterative method application on nonlinear function. In the first regard it is a classical and very knowledge path to resolving without problems same numeric analysis applications. However, we can observe that the asymptotic behaviour of p_n depends strongly on r parameter:

- for $0 \le r < 1$, p_n decrease as n and p_n approach 0;
- for $1 \le r \le 2$, p_n monotonically approaches $1-1/r$;
- for $2 < r \le 3$, p_n approaches $1-1/r$ with oscillations;
- for $3 < r \le 3.449$, p_n is gradually approaches period motion of period 2;
- for $3.449 < r \le 4$, the system become uncontrolled.

The set of attractors of x_n is shown in figure 1.

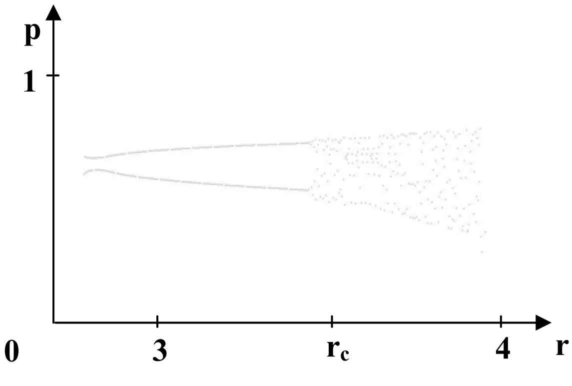

Figure 1. Bifurcation diagram of logistic map for $3 \le r \le 4$

Historically, the logistic map was obtained from the logistic equation, which describes the growth of a population in a closed area:

$$\frac{du}{dt} = (\varepsilon - hu)u \tag{5}$$

If we put this equation into a difference equation form:

$$\frac{u_{n+1} - u_n}{\Delta t} = (\varepsilon - hu_n)u_n \tag{6}$$

we obtain the logistic map if we change the variables as:

$$r = 1 + \varepsilon\Delta t$$
$$p_n = \frac{h\Delta t}{1+\varepsilon\Delta t}u_n \tag{7}$$

The solution of (5) can be obtained analytically for any initial condition $u(0)>0$. It monotonically approaches a fixed point ε/h. By contrast, the difference equation for large interval Δt and the logistic map behave quite differently, producing chaos. This kind of discrepancy between the solution of a differential equation and that of its difference equation appears in any nonlinear system if the difference interval is sufficiently large. Hence we have to be careful when we numerically solve a differential equation by using a difference equation.
If we modify the logistic equation in the form:

$$p_{n+1} = rp_n/(1-p_n), \quad 0 \le r \le 4 \tag{8}$$

we can observe an interesting result about the map equation (figure 2). In this figure we can observe flip-flop behaviour for the 2.3 value of r, in the logistic map diagram. Relation (8) is often used in various probabilistic systems. On this idea the authors speculated in own researched the presented features for some oscillating systems. Another application area of this result can be in the numeric analyses. Here, like as previous problem, is possible to launch a study based on the basin attractor boundary of the specific numerical methods. In this sense can be performed unusual analysis for some numerical techniques from the complexity calculus.

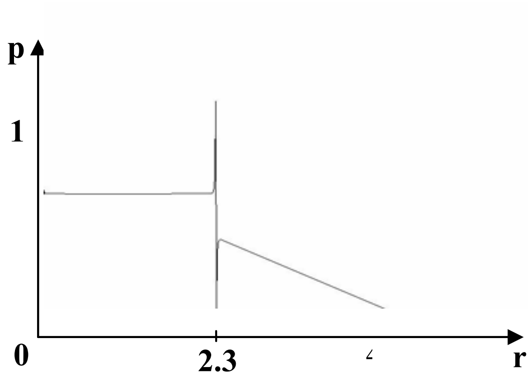

Figure 2. Bifurcation diagram of modified logistic map for $0.01 \le r \le 4$

In the basin attractor studies was obtained in last time some interesting results. Thus, if trying to solve the equation z^4-$1=0$ in the complex plan, we can obtain the Newton's fractal [8,9,10] which shown like in figure 3. And this is not the single case when a numeric method for solving nonlinear algebraic equations has same strange behaviour.

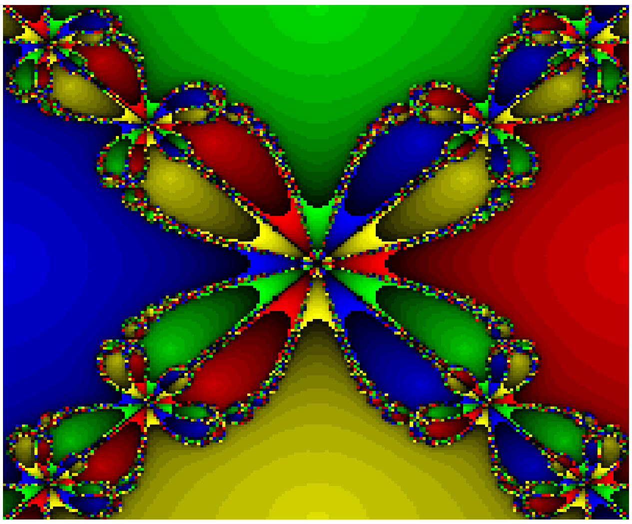

Figure 3. The Newton-Raphson fractal for z^4-1=0 equation

Using fractal basin boundary's theory is possible to orient the research to the pattern recognition problem for the graphical treated systems. In this case, we focused on the fractal basin boundary modelling like a separated boundary between different classes problem.

Fractals by maps

For a given map [5,6]:

$$x_{n+1} = f(x_n) \qquad (9)$$

the set of initial points $\{x_0\}$ whose iterated points never diverge ($|x_n|<\infty$ for any n) is called *Julia set*. For many maps, the *Julia sets* are known to be fractals. A good example is the following complex logistic map:

$$f(z) = az(1-z) \qquad (10)$$

where $z=x+jy, j=\sqrt{-1}$.
In the same way, equation:

$$g(z) = z^2 - b \qquad (11)$$

conducts to another fractal. To set of complex parameters b such that successive iterates of $z=0$ under $g(z)$ do not tend to ∞ is named the *Mandelbrot set*. This set has a fractal border.
When we solve an algebraic equation numerically by Newton's method, we have to iterate a map similar to (11). If the equation have several solutions, an initial value for the iteration will be attracted to one of the solution. The boundary of the set of points that finally converge to one of the solution becomes a fractal. Two initial points that are arbitrarily close can approach distinct solutions, if they start close to this boundary.
Another simple method to construct fractals is provided by *contraction maps*. A contraction map is a mapping that shortens the distance between any two points. It is trivial that the invariant set of a single contraction mao is a point. However, for two or more contraction maps the invariant set is the set X which satisfies:

$$x = f_1(x) \cup f_2(x) \cup \ldots \cup f_n(x) \qquad (12)$$

which is a fractal.

For example, in the case $n=2$, the following maps produce the Cantor set in the [0,1] interval.

$$f_1(x)=\frac{x}{3}, \quad f_2(x)=\frac{2+x}{3} \qquad (13)$$

In the complex plane we have the Koch curve if the mappings are:

$$f_1(z)=\alpha\bar{z}, \quad f_2(z)=(1-\alpha)\bar{z}+\alpha, \\ \alpha=\frac{1}{2}+\frac{\sqrt{3}}{6}j \qquad (14)$$

where $\bar{z}$ denotes the complex conjugate of z.
Thus all regular (non-random) fractals can be expressed in this formalism, which because its simplicity is expected to become more important in future.

Random clusters

Consider a 2 or 3 dimensional lattice and distribute points randomly on it with p probability.If neighbouring sites are occupied by points, they are regarded as connected. By changing the probability p of the occupation of sites we can estimate the critical probability p_c and fractal dimension of clusters.
The fractal dimension of clusters is calculated in the following way. We define the mean radius of clusters of size s as:

$$R_s=<(\sum_{i=1}^{s}\frac{r_i^2}{s})^{1/2}> \qquad (15)$$

where r_i denotes the distance between the centre of mass and the i-th point, and <•> indicates the average over all s-clusters. When R_s is proportional to a power of s, the clusters are statistical fractals with dimension D which satisfies the relation:

$$R_s \sim S^{1/D} \qquad (16)$$

The result of simulations show that (16) holds at $p=p_c$ and the fractal dimensions are estimated as 1.9 (2 dimensional lattice) and 2.5 (3 dimensional lattice) [1]. This value in the 2 dimensional case agrees with the experimental value.
The critical point p_c is known to depend on the type of underlying lattice. On a square lattice $p_c=0.59$, on a honeycomb lattice $p_c=0.70$, and on a triangular lattice $p_c=0.50$.
However, the fractal dimension and other critical indices are anticipated to be universal and independent of the underlying lattice.

Clusters in spin systems

The best-know model of magnetic material is the *Ising model* [7]. In this model, spins which can take only the value +1 or -1 are arranged on a lattice. The total energy (or Hamiltonian) E of the system is given by the equation:

$$E=-J\sum\sum S_iS_j-H\sum_i S_i, \quad S_i=\pm 1 \qquad (17)$$

Here, $\Sigma\Sigma$ denotes summation over nearest neighbour sites. J is the coupling constant and H is the external field. In thermal equilibrium, the probability of occurrence of the state with total energy E is given by:

$$W \sim e^{-E/(k_BT)} \qquad (18)$$

where k_B is Boltzmann's constant and T denotes temperature.
A numerical simulation is performed as follows. First, specify an appropriate initial state, which may be random or uniform. Then, choose one spin at random and calculate the change of total energy of the system assuming that the spin is reversed. Change the sign of the spin according to the probability calculated from (18). Choose another spin at random and repeat the same process. After a large number of repetitions, thermal equilibrium is obtained.
In both 2 and 3 dimensional space, the Ising model is known to show a phase transition at a critical temperature, T_c. For $T<T_c$, symmetry is spontaneously broken and most spins take the same value, which indicates that the system is ferromagnetic. On the other hand when $T>T_c$, each spin takes the value +1 or -1 nearly

independent of neighbouring spins and the average of spin vanishes, which shows that the system is demagnetised. At the critical point $T=T_c$, the characteristic size of clusters of the same spin diverges and distribution of the clusters becomes fractal. The fractal dimension of the clusters is estimated to be 1.88 in 2 dimensional space and 2.43 in 3 dimensions.

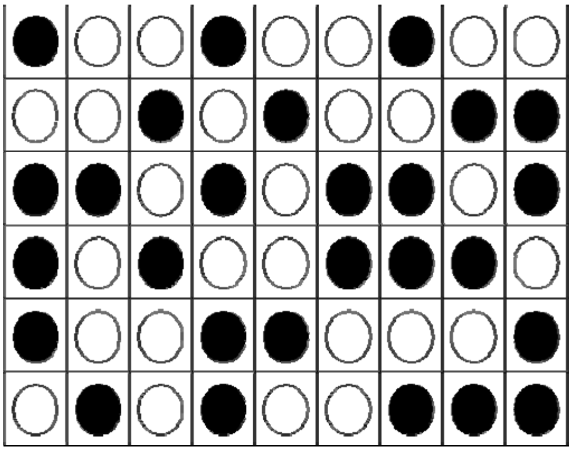

Figure 4. An incompressible mixture of A and B particles

Applications

One interesting application of the chaos theory consists in the stochastic modelling of electric breakdown. We assume that the electric breakdown between a pair of electrodes spreads stochastically with proportional probability to the local electric field. Let Φ denote the electric potential which takes values 1 or 0 at the electrodes. Solve the discrete version of the Laplace equation [1]:

$$\Delta\Phi = \frac{\partial\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y^2} = 0 \qquad (19)$$

on a lattice space with the given boundary conditions. Choose a site at random from the neighbouring sites of the electrode with Φ=1, with proportional probability to the gradient of Φ. The bond connecting this site to the electrode is then regarded as broken and the value of Φ becomes 1 at the site. Solve the Laplace equation with the new boundary condition and again choose at random a neighbouring site of the electrode and its attached cluster, with proportional probability to the gradient of Φ. Repeat the same procedure again and again. We can observe that cluster of sites forms the shape of a lightning discharge. Is proved that the shape is identical to that of DLA (Diffusion Limited Aggregation).

This model has been extended to the more general η-model, in which we choose a broken site with proportional probability to $|\nabla\Phi|^{\eta}$. The basic model is the special case $\eta=1$. When $\eta>1$ the difference of gradients $|\nabla\Phi|$ is enhanced, hence sites near a sharp tip of the cluster are more likely to be broken and we have a cluster with smaller fractal dimension. The fractal dimension of a cluster, which grows from a point-like electrode in d dimensional space, is approximately given by the following:

$$D = \frac{d^2 + \eta}{d + \eta} \qquad (20)$$

Another interesting application of the fractal's results is the self-avoiding random walks. This is a random walks that never intersects its own trajectory. Though this condition is very simple, theoretical treatment becomes extremely difficult, since the whole past trajectory affects the present motion. Self-avoiding random walks are considered as a model of a polymer. Thread-like polymers in solution are self-avoiding entangled by thermal fluctuation. The fractal dimension of self-avoiding random walks in 3 dimensional space is obtained approximately as 5/3, which coincides with the experimental values for polymers. This value 5/3 can also be deduced theoretically by a dimensional analysis.

The theory of complexity provides possibility to produce complicated structures by a simple rule. Considering the fact that any living creature is formed from a finite amount of DNA, the idea of producing complicated structure by simple rules seems promising. The numerical models called *cellular automata* are studied in order to clarify this problem.

A cellular automata has the following five properties:

1. It is defined on a discrete lattice.
2. Time evolution is discrete.
3. The number of states at each site is finite.
4. The rule of evolution is deterministic.
5. The evolution rule is governed by the state of neighbouring sites.

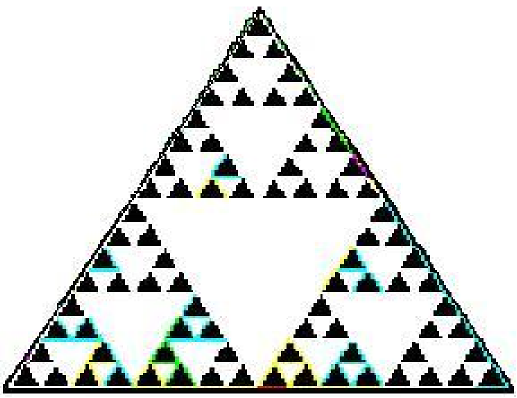

Figure 5. The Sierpinski triangle

Here is the simplest non-trivial example. Let $a_i(n)$ denote the state of the i-th site on a 1 dimensional lattice at time step n. The value taken by $a_i(n)$ is either 1 or 0. The evolution rule is given as:

$$a_i(n) = a_{i-1}(n-1) + a_i(n-1) \bmod 2 \qquad (21)$$

With the initial conditions:

$$a_0(0) = 1, \quad a_i(0) = 0, \quad i \neq 0 \qquad (22)$$

the result pattern is nothing but Sierpinski's gasket in discrete space-time (figure 5).

Conclusions

In the complexity theory is notable involving of the iterative functions in behaviour of fractal pattern. Study of the nonlinear equation, treated into iterative techniques, makes the subject of this paper. It consists in a short revue of the most important principles of the fractal calculus and complexity applications in fundamental sciences and technologies. Were been presented also some new ideas of analysis to iterative relations like as named *the modified logistic equation*, which map is shown in figure 3. On this relation can be performing some studies with valuable results in numeric analysis area.

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