ON A GENERALIZATION OF FOURIER INTEGRAL TRANSFORM

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Abstract. *In the present paper, we focus on an integral transform that generalizes the classical Fourier transform. The generalization is given by the replacement, in the standard definition, of Lebesgue integral with the so-called Henstock-Kurzweil integral. The Henstock-Kurzweil Fourier transform was introduced in [8], where some basic properties were proved (namely existence results, continuity, differentiability and convolution properties). We obtain two existence criteria of Dirichlet and Abel type, a property related to continuity, and finally, a convergence result involving sequences of convolution products.*
Keywords. *Fourier transform, Henstock-Kurzweil integral, convolution.*

Introduction

There has been a resurgence of interest in integrals of highly oscillating real functions, or, in other words, in integrals of functions with non-absolute convergent integral. In this line, the most appropriate concept is the Henstock-Kurzweil integral, introduced independently by Kurzweil and Henstock in 1957/58. It is a notion that encompasses the Riemann and Lebesgue integrals and it has proved itself useful, in the last twenty years, in the study of differential and integral equations (see [1], [2], [4], [5] etc). Recently, Talvila ([8]) presented an application of Henstock-Kurzweil integral to Fourier transform. In this paper, following the line of [8], we obtain some new results on the HK-Fourier transform.

Henstock-Kurzweil integral

Let us begin by recalling its definition in the case of a bounded interval (see, e.g. [3]).

Definition 1. A function $f:[a,b]\to\Re$ is said to be Henstock-Kurzweil (shortly, HK-) integrable if there exists a real number I such that, for every positive ε, one can find a positive function δ_ε on [a,b] (named a gauge) with the property that, for any δ_ε-fine partition $(I_i,t_i)_{i=1}^n$ of [a,b] (that is to say that, for each i, $I_i\subseteq(t_i-\delta_\varepsilon(t_i),t_i+\delta_\varepsilon(t_i))$), one has

$$\left|\sum_{i=1}^n f(t_i)m(I_i)-I\right|<\varepsilon.$$

The real I is denoted by $(\mathrm{HK})\int_a^b f(x)dx$.

In the unbounded case, the Henstock-Kurzweil integral is defined as follows (see [7]):

Definition 2. Let $f:[-\infty,\infty]\to\Re$. A gauge is a (multi)function δ that associates to any $t\in[-\infty,\infty]$ an open interval (that is, an interval of the form $(a,b),[-\infty,b),(a,\infty],[-\infty,\infty]$) which contains t. A partition $(I_i,t_i)_{i=1}^n$ of $[-\infty,\infty]$ is δ-fine if $I_i\subseteq\delta(t_i)$ for each i. With the convention that the measure of an unbounded interval is 0, we say that *f* is Henstock-Kurzweil-integrable if there exists a real number I such that, for every positive ε, one can find a gauge δ_ε such that, for any δ_ε-fine partition of $[-\infty,\infty]$, one has

$$\left|\sum_{i=1}^n f(t_i)m(I_i)-I\right|<\varepsilon.$$

One can prove (Theorem 1 in [7]) that f is HK-integrable (the integral being I) if and only if it is integrable on each compact interval $[a,b]\subseteq[-\infty,\infty]$ and there exists the limit

$$\lim_{\substack{a\to-\infty\\ b\to\infty}}(HK)\int_a^b f(x)dx=I.$$

This notion of integral is strictly more general than those of Riemann, of Lebesgue, as well as the Riemann and Lebesgue improper integrals. Moreover, its natural definition (by means of integral sums) is much more easily to hand. A very simple example of HK-integrable function which is not Lebesgue (therefore, nor Riemann) integrable is given by the derivative of function $F:[0,1]\to\Re$ defined by $F(x)=x^2\sin x^{-2}$ if $x\neq 0$ and $F(0)=0$.

It is not difficult to see (from the completeness of the space of real numbers) that the following Cauchy-type criterion holds:

Proposition 1. A function $f:[a,\infty)\to\Re$ is HK-integrable if and only if, for every $\varepsilon>0$, there exists $A_\varepsilon>a$ such that, for every $A_2>A_1>A_\varepsilon$,

$$\left|(HK)\int_{A_1}^{A_2} f(x)dx\right|<\varepsilon.$$

One can provide the vector space of HK-integrable function with the topology of the Alexiewicz norm $\|f\|=\sup_{a,b}\left|(\mathrm{HK})\int_a^b f(x)dx\right|$.

We remind the reader of the fact that the multiplication by an (essentially) bounded function does not change the integrability in Lebesgue sense. Concerning the multipliers for HK-integral, we need the following

Definition 3. A function $g:\Re\to\Re$ is said to be of bounded variation if its total variation

$$Vg=\sup\left\{\sum_{i=1}^{n}\left|g(t_i)-g(s_i)\right|\right\}$$

is finite, where the supremmum is taken over all non-overlapping real intervals.

It was proved (e.g. [3]) that, on a compact interval, the product of an HK-integrable function and a bounded variation function still is HK-integrable. As the following result asserts, this remains valid in the unbounded case.

Proposition 2. Let $f,g:[a,\infty)\to\Re$ be two functions, f HK-integrable and g of bounded variation. Then fg is HK-integrable.

Proof. Consider $A>a$. On the interval $[a,A]$, the product fg is HK-integrable. Accordingly to Proposition 1, it suffices to show that for every $\varepsilon>0$, there exists $A_\varepsilon>a$ such that, whenever $A_2>A_1>A_\varepsilon$,

$$\left|(HK)\int_{A_1}^{A_2} f(x)g(x)dx\right|<\varepsilon.$$

Fix then $\varepsilon>0$. By hypothesis and the previously mentioned Proposition 1, denoting by Vg the total variation of function g, there is $A_\varepsilon>a$ such that, for all $A_2>A_1>A_\varepsilon$,

$$\left|(HK)\int_{A_1}^{A_2} f(x)dx\right|<\frac{\varepsilon}{\inf_{x\in[A_1,A_2]}\left|g(x)\right|+Vg}.$$

Hence, $\left|(HK)\int_{A_1}^{A_2} f(x)g(x)dx\right|$

$$<\left\|f\chi_{[A_1,A_2]}\right\|\left(\inf_{x\in[A_1,A_2]}\left|g(x)\right|+Vg\right)<\varepsilon.$$

Henstock-Kurzweil Fourier transform

Let us remind the reader of the definition of HK-Fourier transform, that was introduced in [8].

Definition 4. Let $f:\Re\to\Re$. The HK-Fourier transform of f is the function defined by

$$\hat{f}(\omega)=(HK)\int_{-\infty}^{\infty} e^{-i\omega x}f(x)dx$$

$$=(HK)\int_{-\infty}^{\infty}\cos(\omega x)f(x)dx-i(HK)\int_{-\infty}^{\infty}\sin(\omega x)f(x)dx$$

The first question is whether (or in which conditions) the HK-Fourier transform make sense. As the function $x\mapsto e^{-i\omega x}$ is not of bounded variation on the real line unless $\omega=0$, one cannot claim (as in the classical case) that the HK-Fourier transform exists for any HK-integrable function. It was proved (Proposition 2 in [8]) the following

Proposition 3. Let $f:\Re\to\Re$. Then its HK-Fourier transform can be defined if f is locally HK-integrable (which means that it is HK-integrable on each compact interval) and if $|f|$ is integrable in a neighborhood of infinity or if f is of bounded variation in a neighborhood of infinity with limit 0 at infinity.

We give, in the sequel, two criteria for existence of HK-Fourier transform, inspired by Dirichlet, resp. Abel criteria for existence of Riemann improper integral:

Theorem 1. Let $f, g : \Re \to \Re$, f locally HK-integrable and g of bounded variation.

a) Suppose that one can find two positives K, M such that, for every $A_2 > A_1 > M$ and every $A_1 < A_2 < -M$,

$$\left|(HK)\int_{A_1}^{A_2} e^{-i\omega x} f(x)dx\right| < K$$

and suppose that g has limit 0 at infinity. Then the HK-Fourier transform of the product fg exists at ω.

b) Suppose that f admits HK-Fourier transform at ω. Then the HK-Fourier transform of the product fg exists at ω.

Proof. a) We show that we are able to apply Proposition 1.

Indeed,

$$\left|(HK)\int_{A_1}^{A_2} e^{-i\omega x} f(x)g(x)dx\right| < \left|(HK)\int_{A_1}^{A_2} e^{-i\omega x} f(x)dx\right| \cdot\left(\inf_{x\in[A_1,A_2]}|g(x)| + Vg([A_1, A_2])\right) <$$

$$< K\left(\inf_{x\in[A_1,A_2]}|g(x)| + Vg([A_1, A_2])\right),$$

therefore it can be done lower than a given $\varepsilon > 0$ for A_1, A_2 sufficiently big.

b) it immediately follows from Proposition 2.

Another problem to study about an integral transform is the continuity. Again unlike the classical case, the HK-Fourier transform is not necessarily a continuous function. Instead of justifying this, we prefer to present an example.

Example. Consider

$$f : \Re \to \Re,\ f(x) = \frac{x^3}{x^4 + 3}$$

and calculate the complex integral that gives its HK-Fourier transform.

If $\omega > 0$, then $\hat{f}(\omega) = (HK)\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx$ is the conjugate of $(HK)\int_{-\infty}^{\infty} e^{i\omega x} f(x)dx$ which equals

$$2\pi i \operatorname{Re} z\left(e^{i\omega x}\frac{x^3}{x^4+3}, i\right) = \frac{\pi i e^{-\omega}}{2},$$

by Reziduu Theorem. Consequently, $\hat{f}(\omega) = \frac{\pi i e^{-\omega}}{2}$ if $\omega > 0$ and analogously can be proved that $\hat{f}(\omega) = \frac{\pi i e^{\omega}}{2}$ if $\omega < 0$. In $\omega = 0$ the HK-Fourier transform of our function cannot be defined.

Related to continuity, we can establish the following property:

Theorem 2. Let $f : \Re \to \Re$ be a locally HK-integrable function and $\omega \in \Re$. Then, for any $A_1, A_2 \in \Re$,

$$\lim_{n\to\infty}(HK)\int_{A_1}^{A_2} e^{-i(\omega+\delta_n)x} f(x)dx = (HK)\int_{A_1}^{A_2} e^{-i\omega x} f(x)dx$$

for any sequence $(\delta_n)_n$ convergent to 0.

Proof. First of all, since f is locally HK-integrable and the functions $x \mapsto e^{-i\omega x}$ and $x \mapsto e^{-i(\omega+\delta_n)x}$ are of bounded variation on every compact interval, both HK-integrals in the previous formula exist.

In order to prove the equality, we use a convergence theorem for HK-integral (see [6]). This result implies that if $(g_n)_n$ are real functions such that the sequence of their total variations is bounded, pointwisely convergent to a function g, then

$$\lim_{n\to\infty}(HK)\int f(x)g_n(x)dx = (HK)\int f(x)g(x)dx .$$

We are able to apply it for $g_n(x) = e^{-i(\omega+\delta_n)x}$ which satisfies the inequalities $Vg_n \le M$, for all n and converges to $g(x) = e^{-i\omega x}$. Here M is the double of the integer part of $\frac{(A_2 - A_1)\omega}{\pi}$ plus 1.

Indeed,

$$Vg_n = V(\cos(\omega+\delta_n)x) + V(\sin(\omega+\delta_n)x)$$
$$\le 2\left[\operatorname{int}\frac{(A_2-A_1)(\omega+\delta_n)}{2\pi}+1\right] \le 2\left[\operatorname{int}\frac{(A_2-A_1)\omega}{\pi}+1\right]$$

since one can suppose that $\delta_n \le \omega$ for each n.

We deduce that

$$\lim_{n\to\infty}(HK)\int_{A_1}^{A_2} e^{-i(\omega+\delta_n)x} f(x)dx = (HK)\int_{A_1}^{A_2} e^{-i\omega x} f(x)dx$$

In the same way as in the classical case, the convolution product can be defined (see [8] for more about this subject).

In what follows, we obtain, as consequence of a convergence result in HK integration (Theorem 3.1 in [6]), a new property on sequences of convolution products:

Theorem 3. Let $(f_n)_n$ and f be real functions defined on the real line for which the HK-Fourier transforms exist, such that

$$\left\| e^{-i\omega\cdot}(f_n(\cdot) - f(\cdot)) \right\| \to 0 \text{ as } n\to\infty.$$

Then, for every sequence $(g_n)_n$ of uniform bounded variation, pointwisely convergent to a function g,

$$\hat{f}_n * \hat{g}_n \to \hat{f} * \hat{g}.$$

Proof. Since the HK-Fourier transforms of the functions $(f_n)_n$ and f exist, we can define

$$F_n(t) = (HK)\int_{-\infty}^{t} e^{-i\omega x} f_n(x)dx$$

and

$$F(t) = (HK)\int_{-\infty}^{t} e^{-i\omega x} f(x)dx.$$

By definition of Alexiewicz norm, to impose the condition $\left\| e^{-i\omega\cdot}(f_n(\cdot) - f(\cdot)) \right\| \to 0$ is equivalent to say that the sequence $(F_n)_n$ converges to F uniformly. Therefore, we can apply Theorem 3.1 in [6] in order to obtain that $(HK)\int_{-\infty}^{\infty} e^{-i\omega x} f_n(x) g_n(x)dx$ converges to $(HK)\int_{-\infty}^{\infty} e^{-i\omega x} f(x)g(x)dx$ as $n\to\infty$. This can be written as $(\widehat{f_n g_n}) \to (\widehat{fg})$ or, in other words,

$$\hat{f}_n * \hat{g}_n \to \hat{f} * \hat{g}.$$

One can deduce the following

Corollary. Let $(f_n)_n$ and f be real functions defined on the real line for which the HK-Fourier transforms exist, such that $\left\| e^{-i\omega\cdot}(f_n(\cdot) - f(\cdot)) \right\| \to 0$ as $n\to\infty$.

Then, for every function g of bounded variation,

$$\hat{f}_n * \hat{g} \to \hat{f} * \hat{g}.$$

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