Correct Application of the Discrete Fourier Transform in Harmonics

V. MAIER, S. Gh. PAVEL, C. D. MAIER, I. BIROU
Technical University of Cluj-Napoca / Electrical Power Systems Department, Cluj-Napoca, Romania

***Abstract*—The importance of the harmonic analysis and its frequent utilization in different applications impose a complete accuracy. The conditions which provide a correct harmonic analysis are emphasized and analytically justified in this paper. By using an efficient algorithm, a virtual instrument for the harmonics analysis has been realised in LabVIEW by the authors. This program was a very useful and efficient tool in order to emphasize, from the practical point of view, all the errors analytically outlined. The translation of some superior harmonics to inferior order ones represents a possible and harmful phenomenon pointed out as well.**

Index Terms— **LabVIEW, discrete for Fourier transform**

I. Introduction

The main aim of a sampled wave harmonic analysis is the identification of both amplitude and phase for each harmonic up to an established maximum or possible order. The correct determination of the harmonics characteristics and of as many as possible harmonics is very important for the specificity of each end-user energy consumption, for the power quality assessment and for the perturbations emphasizing either produced by the end-users or by the power utility.

The authors concern to apply the Virtual Instrumentation to the power quality assessment has led to the realization of some LabVIEW applications; among these ones, we can mention REGIDE.vi, used for the harmonics analysis.

During the realization and the improvement of the mentioned program, based on the Discrete Fourier Transform (DFT), several traps have been pointed out.

This paper is focused on the presentation of the possible errors and of the prevention methods of these ones by both the analytical demonstrations and the harmonics analysis program appliance.

II. Methodological Basics

A. Fourier Development for Sampled Functions

The numerical processing of the waves and the analogical - digital conversion have led to the voltages and currents signals scanning, according to the graphical representation from Figure 1 where the $\mathbf{Y_k}$ samples are corresponding to the $\mathbf{y_a(t)}$ values of the analyzed function at the $\mathbf{t_k}$ moments. It can be noticed that the **T** period is divided in **2p** equal parts, so that the $\mathbf{t_k}$ moments are given by the relationship

$$t_k = k \cdot \frac{T}{2p},\ k \in \{0, 2p-1\} \tag{1}$$

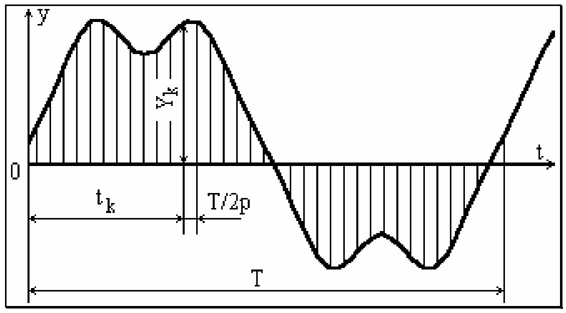

Figure 1. Series of scans, during a period, according to the analyzed wave.

and the correspondents Y_k samples are

$$Y_k,\ k \in \{0, 1, 2, ..., (2p-1)\}. \tag{2}$$

The Fourier coefficients are correctly calculated with the relationships

$$A_N = \frac{1}{p} \cdot \sum_{k=0}^{2p-1} Y_k \cdot sin\left(Nk\frac{\pi}{p} \right),\ N \ge 1; \tag{3}$$

$$B_N = \frac{1}{p} \cdot \sum_{k=0}^{2p-1} Y_k \cdot cos\left(Nk\frac{\pi}{p} \right),\ N \ge 1; \tag{4}$$

$$Y_0 = \frac{1}{2p} \cdot \sum_{k=0}^{2p-1} Y_k\ , \tag{5}$$

where $\mathbf{Y_0}$, representing the arithmetic mean of the $\mathbf{Y_k}$ samples values from a period **T,** is called the **continuous component** of the analyzed wave.

The harmonic analysis of the scanned waves supposes the determination of the Fourier development corresponding to the expression

$$y(t) = Y_0 + \sum_{N=1}^{p-1} (A_N \cdot sin\, Nx + B_N \cdot cos\, Nx), \tag{6}$$

where **x=2pt/T**; the expression (6) emphasizes that only **(p-1)** harmonics can be correctly determined from the **(2p)** pairs of values $(\mathbf{t_k, Y_k})$ and the relationships (3) and (4) point out the necessity that the first argument of the modulation functions, sin (Nkπ/p) and cos (Nkπ/p), has to be nul (the correct initial value of k is 0).

The Fourier development of a pulsing variable, corresponding to relationship (6), can be written in a more condensed form, as follows:

$$y(x) = Y_0 + \sum_{N=1}^{p-1} Y_N \cdot sin(Nx - \varphi_N), \quad (7)$$

where the $\mathbf{Y_N}$ amplitude of a harmonic is

$$Y_N = \sqrt{A_N^2 + B_N^2}\ . \quad (8)$$

B. Phase Identification

For a correct determination of the j_N phase of the **N** order harmonic, the following aspects have to be taken into account:

- the phase definition is included in relationship (7);
- the determined phases are $j_N \geq 0$, situated in the interval [0, 2π];
- the j_N calculus has sense only for $\mathbf{Y_N} \neq 0$;
- in the programming languages (LabVIEW, C, Pascal), **arctg(x)** takes values only in the interval (-π/2, +π/2).

Consequently, when calculating the phases φ_N the A_N şi B_N coefficients signs have to be taken into account in order to place the phases in the correct quadrant. Further on, the next algorithm of phases identification is proposed, being already verified using a realised programme [42]:

- if A_N <0,

$$\varphi_N = \pi + arctg\left(-\frac{B_N}{A_N}\right); \quad (9)$$

- if A_N =0 (so $B_N \neq 0$),

$$\varphi_N = \left(2 + \frac{B_N}{|B_N|}\right) \cdot \frac{\pi}{2}; \quad (10)$$

- if A_N >0,

$$\varphi_N = \left(1 + \frac{B_N}{|B_N|}\right) \cdot \pi + arctg\left(-\frac{B_N}{A_N}\right), \quad (11)$$

with the particular case of $\varphi_N = 0$, if in addition B_N =0. In the relationships (10) şi (11), the notation $|B_N|$ means the absolute value of B_N.

It has to be remarked that the complete identification of a N order harmonic can be done either by the Fourier coefficients (A_N, B_N), or by the pair amplitude - phase (Y_N, j_N).

III. Aspects Of DFT Correct Application

A. The Amplitude Invariance and the Phase Error

In order to realize the analytical motivation of the values range of the addition index, the N order harmonic is taken into consideration written in the general form

$$y_N(x) = Y_N \cdot sin(Nx - \varphi_N), \quad (12)$$

as it appears in relationship (7), for which 2p scans are known on the interval x∈[0, 2π) and having the values

$$y_{Nk} = Y_N \cdot sin\left[N(k-1)\frac{\pi}{p} - \varphi_N\right],\ k = \overline{1, 2p}; \quad (13)$$

the last notation specifies that the natural number k takes values from 1 to 2p.

The calculus of the Fourier coefficients for the considered harmonic is also made with the relationships (3) and (4) with the difference that the addition index is $k = \overline{1, 2p}$ (like in some references) and not $k = \overline{0, (2p-1)}$ as it appears in the referred relationships.

The initial expressions of the calculus relationships are:

$$A_N = \frac{1}{p} \cdot \sum_{k=1}^{2p} Y_N \cdot sin\left[N(k-1)\frac{\pi}{p} - \varphi_N\right] \cdot sin\left(N\frac{k\pi}{p}\right); \quad (14)$$

$$B_N = \frac{1}{p} \cdot \sum_{k=1}^{2p} Y_N \cdot sin\left[N(k-1)\frac{\pi}{p} - \varphi_N\right] \cdot cos\left(N\frac{k\pi}{p}\right); \quad (15)$$

using elementary transformations and formula such as

$$\sum_{k=0}^{n} sin(kx) = \frac{sin\frac{nx}{2} \cdot sin\left(\frac{n+1}{2}x\right)}{sin\frac{x}{2}}; \quad (16)$$

$$\sum_{k=0}^{n} cos(kx) = \frac{cos\frac{nx}{2} \cdot sin\left(\frac{n+1}{2}x\right)}{sin\frac{x}{2}}, \quad (17)$$

we obtain the expressions

$$A_N = Y_N \cdot cos\left(N\frac{\pi}{p} + \varphi_N\right); \quad (18)$$

$$B_N = -Y_N \cdot sin\left(N\frac{\pi}{p} + \varphi_N\right). \quad (19)$$

First of all, owing to relationship (8), it can be remarked the amplitude value $\mathbf{Y_k}$ invariance, which is not depending on the addition index **k** interval, if this one takes **2p** distinct values.

Secondly, it can be noticed that even the same series of values like in the relationships (14) and (15) is described through the way of y_{Nk} samples generation (rel. 13), the phase calculated is greater than the φ_N phase with

$$\Delta\varphi_N = \frac{N\pi}{p}, \quad (20)$$

which is representing a phase calculus error, depending on the p number, being inverse proportional with it. The explanation of the phase error appearance comes from the series of values of the modulation functions $sin(Nk\frac{\pi}{p})$ and $cos(Nk\frac{\pi}{p})$, distinct in the case when $k=\overline{1,2p}$ than in the $k=\overline{0,(2p-1)}$ case.

In conclusion, the phase error does not appear if the relationships (3) and (4) are used for the harmonics identification. If, for the same goal, we use the relationships (14) and (15) then the phase correction is imposed, with the quantity given by (20), the correct phase being

$$\varphi_N' = \varphi_N - \Delta\varphi_N \,, \tag{21}$$

where φ_N represents the phase determined from the A_N and B_N coefficients, calculated with the relationships (18) respectively (19).

The phase error influence is emphasized in Figure 2, for a non-sinusoidal pulsing variable YA, with a known harmonic composition; the samples determined for a big enough number of divisions (2p) have been introduced in the calculus in order that all the harmonics may be identifiable.

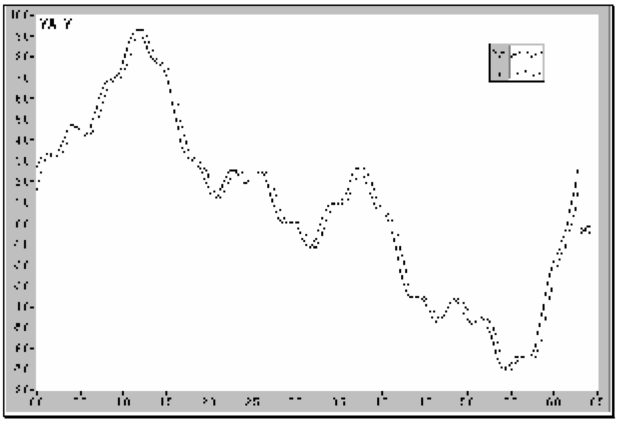

Figure 2. The waves corresponding to the real, analyzed function **YA** and reconstituted one **Y**, using the Fourier analysis.

The harmonic analysis based on relationships as (14) and (15) and the Y wave reconstitution from the identified harmonics, lead to a lagging behind of the reconstituted wave Y versus the analyzed one YA. By correcting the harmonics phases according to relation (21), the reconstituted wave is superposing the analyzed one.

B. Harmonics Maximum Order

The divisions number (2p) is important, first of all, for the maximum order of the harmonic that can be correctly determined which is

$$N_{max} = p-1, \tag{22}$$

contrary to the opinion that p harmonics can be determined. The first argument is that from the (2p) initial conditions (the abscise-ordinate correspondences) the continuous component and 2(p-1) coefficients for (p-1) harmonics are determined, so totally (2p-1) data, meanwhile for p harmonics have to be calculated (2p+1) data, the initial conditions being insufficient. The relationship (2) emphasizes the idea to reconstitute the wave only from (p-1) harmonics, correctly determinable and from the continuous component, when this one exists.

According to the Shannon sampling theorem, the scanning of a non-sinusoidal, periodical signal has to be done with a frequency f_e at least two times greater than the frequency f_{max} of the maximum order harmonic that will be emphasized after the spectral analysis of the signal:

$$f_e \geq 2 f_{max} \,. \tag{23}$$

The frequency representing half of the sampling frequency (f_e/2) is called **Nyquist frequency** [3], having the order p.

If the maximum frequency related to the fundamental frequency

$$f_{max} = N_{max} f_1 \,, \tag{24}$$

is introduced in the relationship (23), a form of the Shannon theorem will be obtained, indicating the maximum order of the harmonic correctly determinable for the sampled wave, as

$$N_{max} \leq p \,. \tag{25}$$

The sign ″=”, allowed by the relationship (25) is disputable, regarding the number of the existing distinct conditions.

For the moment, it can be presumed that just (**p-1**) harmonics can be determined, so **2(p-1)** data. If the continuous component is added, it means that just (2p-1) conditions have been taken into account, one of them remaining available. It can be admitted that this condition may offer a clue about the p order harmonic, but it will not permit a complete identification of this one.

For the analytically study of the particular identification of the Nyquist frequency harmonic, the Fourier coefficients, given by the relationships (3) and (4), are calculated for N=p. The following series of samples is considered

$$Y_{pk} = Y_p \, sin\left(p\frac{k\pi}{p} - \varphi_p \right) = Y_p \, sin(k\pi - \varphi_p), \tag{26}$$

with $k=\overline{0,2p-1}$, for which the Fourier coefficients are resulting as follows:

$$A_p = \frac{1}{p}\sum_{k=0}^{2p-1} Y_p \, sin(k\pi - \varphi_p)\, sin k\pi = 0 \,; \tag{27}$$

$$B_p = \frac{1}{p}\sum_{k=0}^{2p-1} Y_p \, sin(k\pi - \varphi_p)\, cos k\pi = -2Y_p \, sin\varphi_p \,, \tag{28}$$

showing that only $B_p \neq 0$ (A_p=0), for the $\varphi_p \notin \{0,\pi\}$ case, but neither the amplitude nor the phase cannot be calculated. Moreover, it can be noticed that the B_p coefficient (rel. 28) is two times greater than the correct value (rel 19 with the phase correction already introduced).

In conclusion, the Nyquist frequency harmonic is not determinable, being possible utmost to point out that it exists, if casually its phase is different from zero or π.

C. Translation of the Undetectable Harmonics

A special aspect, detected with an original program called REGIDE, is represented by the identification as an inferior order harmonic $K < p$ of a harmonic that has in reality the M order, greater than the detectable limit $M > p$. The results of the program successive processing, with a variable number of samples on a semi-period, in order to identify the wave $y_A = 5\sin(17x+\pi/3)$, is presented in Table 1

TABLE I. IDENTIFICATION OF A GIVEN ORDER HARMONIC (N=17) VERSUS THE SAMPLES NUMBER ON A PERIOD

Samples nr / semi-period NP=p	Identifiable harmonic order (K)	Amplitude (C)	Observations
10	3	5,00	Incorrect harmonic order, amplitude correctly determined.
11	5		
12	7		
13	9		
14	11		
15	13		
16	15		
17	17	8,66	Order correctly identified, incorrect amplitude.
[3] 18	17	5,00	Correct identification.

It can be noticed that even the analyzed wave presents the 17 order harmonic, this one is identified to be as inferior orders $K \in \{3, 5, \ldots, 15\}$ when the number of samples on a semi-period takes respectively the values $p \in \{10, 11, \ldots, 16\}$. The amplitude is however correctly identified.

On the other hand, the amplitude is incorrectly identified when the harmonic order is equal to the number of samples / semi-period (N=p), that confirms one of the previous observations.

Finally, for $p \geq 18 > N = 17$, the identification is correctly realized.

The identification phenomenon of a superior undetectable harmonic ("invisible"), as an inferior harmonic was called the undetectable harmonics translation. This was also emphasized for a wave with a more complex harmonic composition, when the translation phenomenon of an undetectable harmonic is accompanied by the combination of the translated wave with the inferior order harmonic (K<p), which is really existing in the analyzed wave.

The analytical justification of the alias appearance possibility and of its behavior is based on the consideration of the samples series of an $M>p$ order harmonic, as

$$Y_{Mk} = Y_M \sin\left(M\frac{k\pi}{p} - \varphi_M\right),\ k = \overline{0, 2p-1}, \tag{29}$$

for which the problem is if it may appear, after a discrete Fourier analysis, as an N order harmonic. Therefore, the Fourier coefficients A_N and B_N are calculated with the known relationships (3) and (4), but for the samples series given by (29).

Even it is known that the A_N şi B_N coefficients are null for an harmonic with the order $M \neq N$ and $M<N$, the particular cases when the coefficients are not null are searched. After elementary transformations, we obtain the coefficients as follows:

$$A_N = \frac{Y_M}{2p}\left\{\left[\sum_{k=0}^{2p-1}\cos\frac{(M-N)k\pi}{p} - \sum_{k=0}^{2p-1}\cos\frac{(M+N)k\pi}{p}\right]\cos\varphi_M + \left[\sum_{k=0}^{2p-1}\sin\frac{(M-N)k\pi}{p} - \sum_{k=0}^{2p-1}\sin\frac{(M+N)k\pi}{p}\right]\sin\varphi_M\right\}; \tag{30}$$

$$B_N = \frac{Y_M}{2p}\left\{\left[\sum_{k=0}^{2p-1}\sin\frac{(M-N)k\pi}{p} + \sum_{k=0}^{2p-1}\sin\frac{(M+N)k\pi}{p}\right]\cos\varphi_M - \left[\sum_{k=0}^{2p-1}\cos\frac{(M-N)k\pi}{p} + \sum_{k=0}^{2p-1}\cos\frac{(M+N)k\pi}{p}\right]\sin\varphi_M\right\} \tag{31}$$

Using transformations in accordance with (16) and (17), the following sums are resulting null for certain integer numbers *J*, not divisible by *p*:

$$\sum_{k=0}^{2p-1}\sin\frac{Jk\pi}{p} = 0; \quad \sum_{k=0}^{2p-1}\cos\frac{Jk\pi}{p} = 0\ , \tag{32}$$

so that A_N şi B_N are null for the $M< N$ case inclusively.

But if $M> N$ and $(2p)|(M \pm N)$, the Fourier coefficients, according to (30) and (31), may have one of the following aspects:

- if $(2p)|(M-N)$, i.e. $(M-N)$ is divisible by (2p),

$$A_N = Y_M \cos\varphi_M\ ;\quad B_N = Y_M \sin\varphi_M\ ; \tag{33}$$

- if $(2p)|(M+N)$,

$$A_N = -Y_M \cos\varphi_M\ ;\quad B_N = -Y_M \sin\varphi_M\ , \tag{34}$$

confirming from the analytical point of view the alias effect possibility [52].

The condition that a M>p order harmonic appears as a harmonic of N<p order, when (2p) samples are used on a period of the analyzed wave, is analytical expressed by the relationship

$$(2p)\ |\ (M \pm N), \tag{35}$$

that is the integer number $(M \pm N)$ can be divided by (2p).

A relationship for the determination of the apparent frequency (alias) f_{NA} has been proposed in [6] as the absolute value of the difference between the smallest multiple of the sampling frequency and the real frequency of the respective harmonic:

$$f_{NA} = \left| K_{min} f_e - f_M \right| , \quad (36)$$

where f_e represents the sampling frequency,

f_M - the real frequency of the M order harmonic;

K_{min} - the smallest (closest) integer number that leads to the equality fulfillment.

It comes out that the relationships (35) and (36) are equivalent.

Owing the undetectable harmonics translation phenomenon the DFT correctness of a samples series is questionable. The analysis correctness providing can be made in accordance with the next two methods, succeeding the proposed algorithms, as follows:

- sampling at an increased frequency in order to chose from the complete series of values sub-series with different numbers of values. The differences between the harmonic analysis of two different sub-series could emphasize the existence of some undetectable harmonics in the studied wave composition and the results identity would confirm the determination justness. In the cases when differences occur at the amplitudes of some harmonics, it means that translated superior harmonics exist and their order can be determined with relationships as (35) or (36);
- generating, according to the acquired samples series and an interpolation method, of some series of values with different number of samples and applying the harmonic analysis for each one of these series with the results comparison and consequences as presented bellow.

However, from the practical point of view is more efficient and convenient if, before accessing the data acquisition card, the signals would cross a low-pass filter in order to limit the maximum frequency of the signals that have to be then acquired, so analyzed.

IV. Requirements For Dedicated Programs

The main requirements for an accurate and useful harmonic analysis are the following:

- the correct identification of the amplitudes and phases of all harmonics from the analyzed wave;
- neglecting the harmonics with insignificant amplitudes, being situated in the error range of the determination method;
- the reconstitution of the wave from the identified harmonics, the comparison with the wave that has to be analyzed and the determination of the square mean error between them;
- the possibility of adjusting the equipment (feasible using the Virtual Instrumentation) for waves with known harmonics compositions;
- emphasizing the situations when superior harmonics from the analyzed wave are not determined;
- performing the calculus in a period of time as short as possible in order to use the results in quasi-real time (after utmost a period from the acquisition end).

Beside the harmonic analysis of the signals or of some voltages and currents waves, the problem that appears is to emphasize the fast variations (fluctuations) or the slow ones of the voltage from a discrete values series of its effective value.

V. Final Conclusions

The harmonics limitation requires a relevant harmonic analysis of the sampled waves according to DAQ methods. Using both the analytical methods and the CAD analysis, the following recommendations have been established:

- from (2p) samples of the analyzed wave only (p-1) harmonics, defined each one by amplitude and phase, can be determined;
- the correct determination of the harmonics phases is as much important as the amplitude determination, fact that allows the wave reconstitution from its components;
- utilizing the addition index value from the {0, 2p-1} interval, leads to the correct calculus of the harmonics coefficients and phases;
- the harmonics amplitude determination is invariant versus the summing index limits if this one takes (*2p*) values;
- the Nyquist frequency harmonic cannot be correctly determined, but its presence can be emphasized if its initial phase is not null;
- the alias phenomenon being theoretically and practically justified, it requires a special attention in order to prevent the errors.
- any useful program for the harmonics analysis should allow the reconstitution of the wave from the identified harmonics and the comparison with the initial wave. In this way, the square mean error between them can be determined and used as a criterion for the end of the algorithm.

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